Linear extension of the Robinson-Schensted algorithm

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The Robinson-Schensted (RS) algorithm demonstrates a bijection between the set of magnetic configurations f and the set of pairs of tableaux: a semistandard Weyl tableau $P(f)$ accompanied by a standard Young tableau $Q(f)$. We show that it is not only a bijection between sets, but it can be extended to a linear unitary transformation within the space of all quantum states of the magnet.

1 Introduction

In purely combinatorial manner the RS algorithm [1, 2, 3] associates each word f in an alphabet $\tilde{n}$ with a pair $(P(f), Q(f))$ of standard Weyl and Young tableaux [4]. This helps us to find the maximal lengths of nondecreasing (and decreasing) subwords of the word f in the following manner: the maximal length of a nondecreasing subword of f is equal to the length of the top row of $P(f)$ (or $Q(f)$ because these tables have the same shape), similarly, the maximal length of a decreasing subword of f is equal to the height of the first column of $P(f)$ (or $Q(f)$) [5, 6].

In our article we want to show an immediate relation between the RS algorithm and the description of the kinematics of the linear Heisenberg magnetic ring. We will present the quantum relationship between two objects: a magnetic configuration of the ring, and its RS image $(P(f), Q(f))$. We will demonstrate that for the reason of the linear structure of the space $\mathcal{H}$ (the space of all quantum states of the magnet) this relationship is not purely combinatorial but linear. Clearly, the RS algorithm itself defines only a bijective mapping between the two combinatorial sets. In the context of the Heisenberg chain, these two sets have an obvious interpretation of two different orthonormal bases in the space $\mathcal{H}$ of all quantum states of the magnet. This fact stimulates us to extend this algorithm to the space $\mathcal{H}$ by linearity along the general lines described elsewhere [7, 8]. Here we demonstrate explicitly the construction of a wave packet corresponding to the linear version of the RS bijection.

Let's introduce some preliminary facts about Heisenberg magnet, for more detail see [7]. A magnetic configuration of the one-dimensional Heisenberg ring with spin s is the mapping

$$f : \tilde{N} \longmapsto \tilde{n} \tag{1}$$

where $\tilde{N} = \{j = 1, 2, ..., N\}$ - the set of all nodes of the magnet, $\tilde{n} = \{i = 1, 2, ..., n\},\ \ n = 2s + 1$ - the set of a single node spin projections. It can be written in a form

$$|f\rangle = |i_1, i_2, ..., i_N >,\ \ i_j \in \tilde{n},\ \ j \in \tilde{N}. \tag{2}$$

We may treat such a configuration as a word of the length N in the alphabet of spins. We denote by $\tilde{n}^{\tilde{N}}$ the set of all magnetic configurations.

In the next section we describe the duality of Weyl as a physical counterpart of the RS image of a magnetic configuration. Then we construct the corresponding wave packet, illustrate it by an example, and make some final conclusions.

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2 The basis of duality of Weyl

The tableaux $(P(f), Q(f))$ appear in description of the Heisenberg magnet as the labels of the irreducible basis of the Weyl duality. More explicitly, the space of all quantum states of the magnet is

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}}, \tag{3}$$

where $lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}}$ denotes the linear closure of the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations over the field $\mathbb{C}$ of complex numbers. In this space act two groups, symmetric and unitary:

$$A : \Sigma_N \times \mathcal{H} \to \mathcal{H}, \qquad B : U(n) \times \mathcal{H} \to \mathcal{H}. \tag{4}$$

In this way, we get reducible representations in the basis of all magnetic configurations. These two actions mutually commute, it means that appropriate quantities related to both operators ”can be measured simultaneously”. A maximal system of such commuting observables is realized in an irreducible basis in the space $\mathcal{H}$, adopted to the symmetry of both dual groups. So, we need to construct a basis consisting of the vectors which have a specific symmetry of symmetric and unitary groups.

From the theory of representations we know that partitions of the number N label the irreps (irreducible representations) Δ^λ of the symmetric group Σ_N and, at the same time, the irreps D^λ of the unitary group $U(n)$. Consider decomposition of these dual actions into irreps:

$$A = \sum_{\lambda \vdash N, |\lambda| \le n} m(A,\lambda)\ \Delta^\lambda, \qquad B = \sum_{\lambda \vdash N, |\lambda| \le n} m(B,\lambda)\ D^\lambda, \tag{5}$$

on the strength of the Weyl duality [9] we can write:

$$m(A,\lambda) = dim\, D^\lambda, \qquad m(B,\lambda) = dim\, \Delta^\lambda, \tag{6}$$

what means that the multiplicity of an irrep Δ^λ of the symmetric group in the representation A is equal to the dimension of irrep D^λ of the unitary group, and the multiplicity of an irrep D^λ of the unitary group in the representation B is equal to dimension of irreps Δ^λ of the symmetric group, the symbol $|\lambda|$ denotes the number of parts of the partition λ. Eqs. (5,6) give the decomposition of the space $\mathcal{H}$ into sectors:

$$\mathcal{H} = \sum_{\lambda \vdash N, |\lambda| \le n} \oplus\ \mathcal{H}^\lambda, \text{ and } \mathcal{H}^\lambda = U^\lambda \otimes V^\lambda, \tag{7}$$

where U^λ and V^λ are carrier spaces for D^λ and Δ^λ, respectively. We take the set $WT(\lambda, \tilde{n})$ of all semistandard Young tableaux of the shape λ in the alphabet $\tilde{n}$ as a standard basis for U^λ, and refer to its elements as to Weyl tableaux. Similarly, we take the set $SYT(\lambda)$ of all standard Young tableaux in the alphabet $\tilde{N}$ as a standard basis for V^λ. Thus, the vectors of the basis of the Weyl duality have a form $|\lambda\, t,\, y\rangle$ where λ - a partition which labels the irreps of Σ_N and $U(n)$, t - a Weyl tableau, y - a Young tableau.

The RS combinatorial algorithm mentioned at the beginning of this article, provides a way of labelling of the irreducible scheme of the Weyl duality, by magnetic configurations $f' \in \tilde{n}^{\tilde{N}}$ in combinatorial unique way, prescribing to each vector $|\lambda\, t\, y\rangle$ a magnetic configuration f' by the reverse RS algorithm. But each state $|\lambda\, t\, y\rangle$ should have a specified symmetry. So, it is now obvious, that each irreducible state $|\lambda\, t\, y\rangle$ is a definite linear superposition of a number of magnetic configuration. In other words we are looking for a wave packet of the specified symmetry, given by a pair (t, y) of tableaux.

3 Linear version of the RS algorithm

Under the action of the group Σ_N the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations decomposes into orbits $\mathcal{O}_\mu$, where μ denotes the weight of magnetic configuration. Such an orbit carries the transitive representation

$R^{\Sigma_N:\Sigma^\mu}$, where Σ^μ is the Young subgroup $\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \ldots \times \Sigma_{\mu_n}$. This transitive representation decomposes into irreps of Σ_N:

$$R^{\Sigma_N:\Sigma^\mu} \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu}\, \Delta^\lambda. \tag{8}$$

The basis elements of this representation are of the form $|\mu\lambda\, t\, y\rangle$, these are the same basic vectors as those of the duality of Weyl but restricted to an orbit $\mathcal{O}_\mu$, what is indicated by the additional quantum number μ. Elements of this basis can be expressed as a linear combination of some magnetic configuration from the orbit $\mathcal{O}_\mu$

$$|\mu\,\lambda\,t\,y\rangle = \sum_{f\in\mathcal{O}_\mu} \begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix} |f\rangle, \tag{9}$$

where coefficients of this expansion (Kostka matrices at the level of bases [7]) are very closely related to the RS algorithm, and have form

$$\begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix} = \sum \begin{bmatrix} \{1\} & \{1\} & \lambda_{12} \\ f(1) & f(2) & t_{12} \end{bmatrix} \begin{bmatrix} \lambda_{12} & \{1\} & \lambda_{123} \\ t_{12} & f(3) & t_{123} \end{bmatrix} \ldots \begin{bmatrix} \lambda_{1\ldots N-1} & \{1\} & \lambda \\ t_{1\ldots N-1} & f(N) & t \end{bmatrix}, \tag{10}$$

where the sum runs over appropriate labels of intermediate representations. The coefficient

$$\begin{bmatrix} \lambda_{1\ldots j-1} & \{1\} & \lambda_{1\ldots j} \\ t_{1\ldots j-1} & f(j) & t_{1\ldots j} \end{bmatrix} \tag{11}$$

is defined unambiguously by reverse RS algorithm at the steps $N-j$ and $N-j-1$. $t_{1\ldots j}$ is the remaining Weyl tableau at the j-th step of reverse algorithm, $\lambda_{1\ldots j-1}$ its shape, whereas, $t_{1\ldots j-1}$ is the remaining Weyl tableau at the $N-j-1$-th step, $\lambda_{1\ldots j-1}$ its shape, and $f(j)$ is the j-th letter of the configuration $f \in \mathcal{O}_\mu$ [8]. Let's point out, that a pair of tableaux $|t\,y\rangle$ arose from a configuration f', not necessarily equal to f. $|f'\rangle$ has meaning as the word on the alphabet $\tilde{n}$, whereas $|f\rangle$ physically denotes a product of single node states of the Heisenberg magnet. The coefficient $\begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix}$ gives the amplitude of the product state $|f(1)\,f(2)\ldots f(N)\rangle$ in the state $|\mu\,\lambda\,t\,y\rangle = |RS(f')\rangle$ of the duality of Weyl. The intermediate coefficient (11) corresponds to the Littlewood-Richardson decomposition:

$$D^{\lambda_{1\ldots j-1}} \otimes D^{\{1\}} = \sum_{\lambda_{1\ldots j}} \oplus\ D^{\lambda_{1\ldots j}} \tag{12}$$

for the unitary group $U(n)$. And because in this decomposition we can add a box to the diagram $\lambda_{1\ldots j-1}$ only in places admissible by standardness, thus we don't need any repetition labels. In the case of the Heisenberg model with the single node spin $s = 1/2$ (model reveal $SU(2)$ symmetry) the intermediate coefficients (11) reduces to the Clebsch-Gordan coefficients for angular momentum theory. In this case (9) reads

$$|\mu\,\lambda\,t\,y\rangle = \sum_{f\in Q^{(\mu)}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix} \cdot \begin{bmatrix} j_{12} & j_3 & j_{123} \\ m_{12} & m_3 & m_{123} \end{bmatrix} \ldots \begin{bmatrix} j_{12\ldots N} & j_N & J \\ m_{12\ldots N} & m_N & M \end{bmatrix} \cdot |f\rangle \tag{13}$$

where angular momentum $j_{1\ldots k}$ is determined by irrep $\lambda_{1..k}$, and projection of angular momentum $m_{1..k}$ by Weyl tableau $t_{1..k}$, $k = 1\ldots N$. For the Heisenberg magnet with single node spin $s \geq 1/2$, a way of calculating the coefficients (11), one can find in the work of Louck [10].

λ	$\{4\}$	$\{31\}$			$\{2^2\}$	
t	$\boxed{+\,\vert\,+\,\vert\,-\,\vert\,-}$	$\begin{array}{\|c\|c\|c\|}\hline + & + & - \\ \hline - & & \\ \hline\end{array}$			$\begin{array}{\|c\|c\|}\hline + & + \\ \hline - & - \\ \hline\end{array}$	
y	$\boxed{1\,\vert\,2\,\vert\,3\,\vert\,4}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$
f \ f'	$++--$	$-++-$	$+-+-$	$+--+$	$--++$	$-+-+$
$\vert ++--\rangle$	$\frac{1}{\sqrt{(6)}}$	0	$\frac{1}{\sqrt{(3)}}$	$\frac{1}{\sqrt{(6)}}$	$\frac{1}{\sqrt{(3)}}$	0
$\vert -++-\rangle$	$\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{(3)}}$	$\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2\sqrt{(3)}}$	$-\frac{1}{2}$
$\vert +-+-\rangle$	$\frac{1}{\sqrt{(6)}}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{(3)}}$	$\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2\sqrt{(3)}}$	$\frac{1}{2}$
$\vert +--+\rangle$	$\frac{1}{\sqrt{(6)}}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{(3)}}$	$-\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2\sqrt{(3)}}$	$-\frac{1}{2}$
$\vert --++\rangle$	$\frac{1}{\sqrt{(6)}}$	0	$-\frac{1}{\sqrt{(3)}}$	$-\frac{1}{\sqrt{(6)}}$	$\frac{1}{\sqrt{(3)}}$	0
$\vert -+-+\rangle$	$\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{(3)}}$	$-\frac{1}{\sqrt{(6)}}$	$-\frac{1}{2\sqrt{(3)}}$	$\frac{1}{2}$

Table 1 Kostka matrix at a level of bases for $N = 4,\ \ n = 2,\ \ \mu = \{2^2\}$

4 Example

Tab.1 presents the Kostka matrix at the level of bases for $N = 4,\ \ n = 2,\ \ \mu = \{2^2\}$. The columns are labelled by the irreducible basis of the Weyl duality, rows by the magnetic configurations.

5 Conclusions

This algorithm plays two roles in the context of the Heisenberg magnet. Firstly, it serves for labelling the irreducible basis of the Weyl duality, so that $|\lambda\, t\, y\rangle = |RS(f')\rangle$. Secondly, it provides a complete information for construction of $|\lambda\, t\, y\rangle$ in terms of magnetic configuration f, or, more shortly, for evaluation of elements $\langle f|RS(f')\rangle$ of the Kostka matrix at the level of bases. We stress at this point that magnetic configurations f and f' involved in these two roles have conceptually different quantum interpretations. We also record an interesting result that the diagonal coupling coefficient $\langle f|RS(f)\rangle$ is a product of appropriate Wigner-Clebsch-Gordan coefficients, without any summation over intermediate basis, and with no repetition labels.

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