The role of paths in Robinson - Schensted algorithm (RS), and Kerov, Kirillov and Reshetikhin bijection (KKR)

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In this work we want to amplify the RS algorithm in the language of paths. Thus, it is a continuation of the previous work [6], where the magnetic interpretation of RS algorithm was given. We also want to present the intermediative role of paths in bijection between standard Young tableaux and rigged configurations. We will treat the paths as a "bridge" which allow us to move between this two objects.

1 Introduction

The Heisenberg model of magnetism is a good example of applications of combinatoric notions in nanoscopic physics. The solutions of the eigenproblem of the isotropic Heisenberg Hamiltonian in one dimension, i.e. for linear magnetic ring consisting of N nodes, each with the spin 1/2, was given by Bethe [1] in 1931. Nowadays it is known as the Bethe Ansatz (BA). An important ingredient of BA is the hypothesis of strings, which yields a classification of BA eigenfunctions in terms of some combinatorial objects, described by Kerov, Kirillov and Reshetikhin [2] as rigged string configurations.

We intend to combine these objects with the path described by Dasmahapatra [4] in the language of duality of Weyl [5]. We will present the role of the latter in making predictions about character of solutions of the Heisenberg Hamiltonian.

2 Preliminaries

We consider a linear magnetic Heisenberg ring, consisting of N nodes, each with the spin s. This provides us with two sets, the set of nodes of the magnetic ring

$$\tilde{N} = \{j = 1, 2, ..., N\} \tag{1}$$

and the set of all single-node spin projections

$$\tilde{n} = \{i = 1, 2, ..., n\}, \ \ n = 2s + 1. \tag{2}$$

The set $\tilde{n}$, *the alphabet of spins*, delivers an orthonormal basis for the carrier space h of a single-node spin s, such that

$$h = lc_{\mathbb{C}}\tilde{n}, \ \ \dim h = n = 2s + 1. \tag{3}$$

In order to obtain an orthonormal basis for the space $\mathcal{H}$ of all quantum states of the magnet, we define a magnetic configuration on the ring $\tilde{N}$ as

$$f : \tilde{N} \longrightarrow \tilde{n}, \quad |f\rangle = |i_1, i_2, ..., i_N >, \ \ i_j \in \tilde{n}, \ \ j \in \tilde{N}. \tag{4}$$

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path 1	path 2	path 3	path 4
1. [1]	1. [1]	1. [1]	1. [1]
2. [1 2]	2. [1 / 2]	2. [1 2]	2. [1 / 2]
3. [1 2 3]	3. [1 / 2 / 3]	3. [1 2 / 3]	3. [1 3 / 2]
	This path is allowed for $n > 2$		

Table 1 Combinatoric way of creation of irreps of U(n)for N=3.

The set of all magnetic configurations

$$\tilde{n}^{\tilde{N}} = \{f : \tilde{N} \longrightarrow \tilde{n}\} \tag{5}$$

allows us to write the space $\mathcal{H}$ in the form

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}} \cong \prod_{j \in \tilde{N}} \otimes\ h_j, \tag{6}$$

where h_j is a faithful copy of the linear unitary space h, attributed to the node $j \in \tilde{N}$.

At that moment we need some important information about the scheme of Weyl duality [5], but for the reason of conciseness we refer readers to our latest work [6], so we will proceed with the notation established there.

3 The paths

We intend to present the paths considered by Dasmahapatra [4] in terms of a standard Young tableau (Young diagram whose cells are occupied by integers that strictly increase along the rows and along the columns). One path presents formation of one irrep $D^{\{\lambda\}}$ of the unitary group U(n) in decomposition of one of the dual actions B [6]. This fact implies that the total number of paths is equal to $\sum_{\lambda \vdash N, |N| \leq n} m(B, \lambda)$, that is also the number of all standard Young tableaux of shape $\lambda \vdash N$ with $|N| \leq n$.
Let us consider three particles, that is N=3. We create irreps $D^{\{\lambda\}}$ of U(n) for this three particles combinatorically. The number of possible ways of creation of such irreps is equal to the number of paths and thus the number of standard Young tableaux of shape $\lambda \vdash N$ with $|N| \leq n$, as seen in table 1. This process can be also expressed briefly in the form

$$(D^{\{1\}})^{\otimes 3} = D^{\{3\}} \oplus 2D^{\{2\ 1\}} \oplus D^{\{1^3\}}. \tag{7}$$

Following that example, it is not difficult to see that there exists a bijection between paths and standard Young tableaux, what was also discussed in [6] in the context of RS algorithm. If we look carefully at the definition of the element of the Kostka matrices at the level of bases

$$\left[\begin{array}{ccc}\mu & \lambda & t\\ f & y & \end{array}\right] = \sum$$

$$\left[\begin{array}{ccc}\{1\} & \{1\} & \lambda_{12}\\ f(1) & f(2) & t_{12}\end{array}\right]\left[\begin{array}{ccc}\lambda_{12} & \{1\} & \lambda_{123}\\ t_{12} & f(3) & t_{123}\end{array}\right]\ldots\left[\begin{array}{ccc}\lambda_{1\ldots N-1} & \{1\} & \lambda\\ t_{1\ldots N-1} & f(N) & t\end{array}\right], \tag{8}$$

where the sum runs over appropriate labels of intermediate representations, then we find out that the first row of this coefficient depicts the ladder construction, presented in the table 1, equipped with the standard bases, in row two, corresponding to the irreps presented in the first row.
The paths treated as standard Young tableaux naturally eliminate coefficients (8) where it is not possible to insert the elements of the magnetic configuration f in the shape given by the path. That is the role of paths in RS bijection.

The transitive representation of the symmetric group Σ_N

$$R^{\Sigma_N:\Sigma^{\nu}} \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu}\Delta^{\lambda} \tag{9}$$

presents the linear structure of the second dual action A of the symmetric group, where multiplicities $K_{\lambda\mu}$ constitute the Kostka matrix for the group Σ_N and $\lambda \trianglerighteq \mu$ denotes the dominance order [6][7]. Such presentation helps us to observe that for n=2 the state of symmetry given by standard Young tableau is the eigenvector of the total spin S. This fact does not hold for $n > 2$, what was shown in table 2.

4 Bijection between paths and strings

For the reason of conciseness we refer readers to [8] (in this conference) for some ground information about string configuration, and like previously we will continue with the notation established there.
As we told earlier there is a bijection between paths and standard Young tableaux, and between standard Young tableaux and rigged configurations [2], [4], as well. This fact implies the bijection between paths and rigged configuration [4]. Lets restrict ourselfs to the case n=2. The duality of Weyl between symmetric group Σ_N and the unitary group U(2), related to the single-node spin 1/2, allows us to replace the Heisenberg model by the system of r spin deviations, or Bethe pseudoparticles. For each such a system, pseudoparticles move on the ring $\tilde{N}$ by jumps from a site $j \in \tilde{N}$ to a nearest neighbour $j' = (j \pm 1)_{modN}$. The first column of the table 3 contains magnetic configurations, the second - the corresponding Young tableaux under the RS bijection [7], and the fourth - the rigged string configurations resulting from KKR bijection, for the case N=6 and n=2. Some valuable information about eigenstates of the Heisenberg Hamiltonian are provided by string configuration and therefore by the paths because we are able to write down the rigging of the string configuration from the path [4]. These two objects inform us, at the very begining, about the character of the solutions of the Heisenberg Hamiltonian.

Each string has a definite length, given by a row of the string configuration. Strings can move on the nodes unoccupied by other strings, as well as on slopes of longer strings up to reaching the same height. In this way, strings of a given length l have the total number P_l of moves, referred to as "the number of holes of the length l".These holes are distributed either to the left, or to the right of a given string. The number of the former holes is called the riging $0 \leq L_l \leq P_l$ of a given string, and provides an information of the quasimomentum of this string.

5 Final remarks and conclusions

We have shown that the standard Young tableaux naturally exclude existence of some elements of the Kostka matrix at the level of bases.

M			S=4,2,0		S=3,2,1		S=0,2		S=1
4	$R^{\Sigma_4:\Sigma_4}$	=	$\Delta^{\{4\}}$						
3	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$2\Delta^{\{3\ 1\}}$				
2	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$				
2	$R^{\Sigma_4:(\Sigma_2\times\Sigma_2)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$		
1	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$				
1	$R^{\Sigma_4:(\Sigma_2\times\Sigma_1\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$2\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$	+	$\Delta^{\{2\ 1^2\}}$
0	$R^{\Sigma_4:\Sigma_4}$	=	$\Delta^{\{4\}}$						
0	$R^{\Sigma_4:(\Sigma_2\times\Sigma_1\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$2\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$	+	$\Delta^{\{2\ 1^2\}}$
0	$R^{\Sigma_4:(\Sigma_2\times\Sigma_2)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$		
-1	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$				
-1	$R^{\Sigma_4:(\Sigma_2\times\Sigma_1\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$2\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$	+	$\Delta^{\{2\ 1^2\}}$
-2	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$				
-2	$R^{\Sigma_4:(\Sigma_2\times\Sigma_2)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$	+	$\Delta^{\{2^2\}}$		
-3	$R^{\Sigma_4:(\Sigma_3\times\Sigma_1)}$	=	$\Delta^{\{4\}}$	+	$\Delta^{\{3\ 1\}}$				
-4	$R^{\Sigma_4:\Sigma_4}$	=	$\Delta^{\{4\}}$						

Table 2 The decomposition of the transitive representations of Σ_4 on the irreducible representations $\Delta^{\{\lambda\}}$ in case with n=3.

The porpose of this work was also to give a transparent presentation of KKR bijection, which constitutes the essential point in searching for magnetic interpretation of that bijection. Such interpretation will allow us to predict, purely on the combinatorics level, not only the character of the Bethe vectors but the complete set of quantum numbers attributed to that state, as well.

Table 3 shows that we can use paths, created from standard Young tableaux, to obtain the total number of the Bethe vectors. That fact implies that we can search for more information about rigged configurations contained in paths. In particular, the shape of a path derived from a standard Young tableau encodes information on the string configuration and the distribution of quasimomentum over strings.

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Magnetic configuration f	The Young tableau y obtained from f via RSK algorithm	The path obtained from y	String configuration obtained from y via KKR procedure
$--++++$	1 2 5 6 3 4		0 ☐ 2
$-+-+++$	1 3 5 6 2 4		0 0 2
$-++-++$	1 3 4 6 2 5		0 1 2
$+-+-++$	1 2 4 6 3 5		1 1 2
$-+++-+$	1 3 4 5 2 6		0 2 2
$+--+++$	1 2 3 6 4 5		1 ☐ 2
$++-+-+$	1 2 3 5 4 6		2 2 2
$++--++$	1 2 3 4 5 6		2 ☐ 2
$+-++-+$	1 2 4 5 3 6		1 2 2

Table 3 Magnetic configurations, standard Young tableaux, paths and rigged configurations.

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