A combinatoric description of Bethe Ansatz solutions for nanoscopic systems

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Bethe Ansatz provides an exact classification of eigenstates of the Heisenberg Hamiltonian for a finite magnetic ring, consisting of N nodes, each with the spin $s = 1/2$ (with some extensions to an arbitrary spin s) in terms of rigged string configurations. The latter are some combinatorial objects which serve as classification labels for solutions of Bethe equations. An astonishing feature is existence of Robinson-Schensted (RS) and Kerov-Kirillov-Reshetikhin (KKR) bijections between sets of (i) all magnetic configurations, (ii) all pairs of standard Young and Weyl tableaux of N boxes and $n = 2s + 1$ rows, (iii) all rigged string configurations, for given n and s. These bijections allow to point out an exact correspondence between physically admissible solutions of highly nonlinear Bethe Ansatz equations and the initial basis of quantum calculations - magnetic configurations which are just possible distributions of spins over the nodes.

1 Introduction

Exactly soluble models [1-2] play an important role in understanding of physical processes in various complex systems, since they provide, via Bethe Ansatz (BA) [2-4], exact solutions of appropriate equations of motion for systems involving two-body interactions within N elementary objects. A leading example is the linear Heisenberg ring of N spins s, coupled by exchange interactions between nearest neighbours. For, say, $2 < N \lesssim 10^4$, such chains are typical models for nanoscopic systems which supply solutions under full qualitative and quantitative control. Here we aim to point out a somehow surprising role of a relatively simple combinatorics which is able to predict and to classify all solutions of highly non-linear Bethe equations merely on the basis of some combinatorial sorting procedures.

It is well known that the space $\mathcal{H}$ of all quantum states of the linear Heisenberg ring of N spins s is spanned on the set of all magnetic configurations, that is, the set

$$\tilde{n}^{\tilde{N}} = \{f : \tilde{N} \longrightarrow \tilde{n}\} \tag{1}$$

of all mappings $f : \tilde{N} \longrightarrow \tilde{n}$ from the set

$$\tilde{N} = \{j = 1, 2, ..., N\} \tag{2}$$

of all magnetic nodes of the ring, to the set

$$\tilde{n} = \{i = 1, 2, ..., n\}, \;\; n = 2s + 1, \tag{3}$$

labeling all possible z-projections of the single-node spin s such that the single-node state $|i>$, $i \in \tilde{n}$, corresponds to the z-projection $m_i = s - i + 1 \in \{s, s-1, s-2, ..., -s\}$. Just these two sets, $\tilde{N}$ and $\tilde{n}$, give rise to an important combinatorics referred sometimes to as ”combinatorial BA” [5-8], which is capable to classify the eigenstates of the Heisenberg Hamiltonian for the ring $\tilde{N}$, as solutions of appropriate Bethe equations. These two sets constitute also the basis for the famous duality of Weyl [9] between the actions of the symmetric group Σ_N and the unitary group U(n), both acting in the space $\mathcal{H}$ of all quantum states

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of the Heisenberg magnet. We refer in the sequel to the sets $\tilde{N}$ and $\tilde{n}$ as to the alphabet of nodes and spins, respectively.
Here we aim to describe in some detail three sets of combinatorial objects, each of which yields an orthonormal basis in the space $\mathcal{H}$: (i) the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations, (ii) the set of appropriate pairs (t,y) of semistandard Young tableaux, t and y being a tableau in the alphabet $\tilde{n}$ of spins and $\tilde{N}$ of nodes, respectively; this set provides an irreducible basis for the duality of Weyl, (iii) the set of the so called rigged string configurations, which classify eigenstates of the Heisenberg Hamiltonian for the magnetic ring along the Bethe hypothesis of strings; we describe the latter set for s=1/2 only, for reason of conciseness.
Clearly, the most important physical information is enclosed in matrices of linear unitary transformations between these three bases. Problem of explicit determination of these matrices is computationally equivalent to that of solving the eigenproblem of the Heisenberg Hamiltonian, or solving a complete set of appropriate Bethe equations, and thus is clearly outside our actual task. We have to point out, however, that there exist in the literature two remarkable combinatorial bijections between these sets which allow us to classify complete collections of final solutions. The first is the Robinson-Schensted (RS) correspondence [10-11] (cf. also [12-14]) between the first and the second basis in $\mathcal{H}$, whereas the second bijection introduced by Kerov, Kirillov and Reshetikhin (KKR)[5] (cf. also [6-8]), associates each standard Young tableau with a rigged string configuration. Here we present a brief demonstration of these two bijections.

2 Three bases in the space of all quantum states of a finite Heisenberg magnet

We proceed to describe in some detail three orthonormal bases in the space $\mathcal{H}$, mentioned in the introduction. The first is the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations. Each element of this set is a mapping $f : \tilde{N} \longrightarrow \tilde{n}$, which can be presented in the form

$$|f>=|i_1 i_2 ..., i_N>, \;\; i_j \in \tilde{n}, j \in \tilde{N}, \tag{4}$$

or, equivalently, as a *word* of the length N in the alphabet of spins. The unitary structure of the space $\mathcal{H}$ is imposed by

$$< f|f'>= \delta_{ff'}, \;\; f, f' \in \tilde{n}, j \in \tilde{j}. \tag{5}$$

The set $\tilde{n}^{\tilde{N}}$ provides an initial basis for any quantum calculations.

$\mathcal{H}$ is a carrier space of representations of the symmetric group Σ_N and the unitary group U(n),denoted by A and B, respectively. Actions of these groups, defined by

$$A(\sigma) = \begin{pmatrix} f \\ f \circ \sigma^{-1} \end{pmatrix}, \;\; f \in \tilde{n}^{\tilde{N}}, \;\; \sigma \in \Sigma_N, \tag{6}$$

and

$$B(a)|f> \equiv B(a)|i_1 i_2 ... i_N>= \sum_{i'_1 i'_2 ... i'_N} a_{i'_1 i_1} a_{i'_2 i_2} ... a_{i'_N i_N} |f'> \tag{7}$$

$$a \in U(n), \;\; f \in \tilde{n}^{\tilde{N}}, \;\; f' \equiv (i'_1 i'_2 ... i'_N) \in \tilde{n}^{\tilde{N}},$$

give rise to the duality of Weyl. Namely, the commutativity of these two actions,

$$[A(\sigma), B(a)] = 0, \;\; \sigma \in \Sigma_N, \;\; a \in U(n), \tag{8}$$

imposes that the irreducible basis of both actions, $A : \Sigma_N \times \mathcal{H} \longrightarrow \mathcal{H}$ and $B : U(n) \times \mathcal{H} \longrightarrow \mathcal{H}$, yields a complete set of states in the space $\mathcal{H}$ of all quantum states of the magnet, and elements of this set are uniquely classified by labels of irreducible representations and their standard bases. In short, both irreducible representations, Δ^λ entering A and D^λ entering B, can be "measured simultaneously", together with their standard bases, in the meaning of the Heisenberg uncertainty principle.

In order to be more specific, we recall briefly some rudiments of the representation theory of symmetric and unitary groups (cf. e.g.[9,13-14] for more detail). The labels

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \;\; \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0, \;\; \lambda_i \in \mathbb{Z}, \tag{9}$$

are combinatoric objects called *partitions*, and satisfy the constraint $\sum_{i=1}^{n} \lambda_i = N$ and $r \leq n$ for the case of the symmetric group Σ_N and unitary group U(n), respectively. The standard basis for the irrep Δ^λ of Σ_N is the set denoted usually as SYT(λ), consisting of all standard Young tableaux $y \in SYT(\lambda)$ of the shape λ in the alphabet $\tilde{N}$ of nodes. A standard Young tableau $y \in SYT(\lambda)$ consists thus of boxes $y_{\alpha\beta} \in \tilde{N}$, $\alpha = 1, 2, ..., r$, and $\beta = 1, 2, ..., \lambda_\alpha$, such that entries of a Young tableau strictly increase along each row and along each column, that is

$$y_{\alpha\beta} < y_{\alpha\beta'} \quad \text{for} \quad \beta < \beta', \quad \text{each row} \;\; \alpha, \tag{10}$$

$$y_{\alpha\beta} < y_{\alpha'\beta} \quad \text{for} \quad \alpha < \alpha', \quad \text{each column} \;\; \beta. \tag{11}$$

Eqs. (10,11) define the standardness condition for Young tableaux. Dually, the standard basis for the irrep D^λ of U(n), is the set $WT(\lambda, \tilde{n})$ of all semistandard Young tableaux t of the shape λ in the alphabet $\tilde{n}$ of spins, referred hereafter to as Weyl tableaux. Semistandardness of a Weyl tableau $t \in WT(\lambda, \tilde{n})$ reads as

$$t_{\alpha\beta} \leq t_{\alpha\beta'} \quad \text{for} \quad \beta < \beta', \quad \text{each row} \;\; \alpha, \tag{12}$$

$$t_{\alpha\beta} < t_{\alpha'\beta} \quad \text{for} \quad \alpha < \alpha', \quad \text{each column} \;\; \beta, \tag{13}$$

that is, entries of a Weyl tableau weakly increase along each row α, and strictly along each column β of λ.

Thus the second set, that is, the irreducible basis of the Weyl duality for the Heisenberg magnet characterised by the pair (N,n), has the form

$$\begin{aligned} b^W_{irr} = \;\; & \{ |\lambda t y > | \; \lambda \vdash N, \; |\lambda| \leq n, \;\; t \in WT(\lambda, \tilde{n}), \;\; y \in SYT(\lambda) \} = \\ & \textstyle\bigcup_{\lambda \vdash N, |\lambda| \leq n} WT(\lambda, \tilde{n}) \times SYT(\lambda), \end{aligned} \tag{14}$$

where $\lambda \vdash N$ denotes a partition of the integer N, and $|\lambda| = r$ is the number of non-zero parts λ_i of λ. By the definition, we have

$$A(\sigma)|\lambda \, t \, y \,\rangle = \sum_{y' \in SYT(\lambda)} \Delta^\lambda_{y'y}(\sigma) \, |\lambda \, t \, y' \,\rangle, \tag{15}$$

$$B(a)|\lambda \, t \, y \,\rangle = \sum_{t' \in WT(\lambda, \tilde{n})} D^\lambda_{t't}(a) \, |\lambda \, t' \, y \,\rangle, \tag{16}$$

where $\Delta^\lambda_{y'y}(\sigma)$ and $D^\lambda_{t't}(u)$ are elements of standard Wigner matrices for irreps Δ^λ for $\sigma \in \Sigma_N$ and D^λ for $a \in U(n)$, respectively. In this way, the duality of Weyl yields a decomposition of the space $\mathcal{H}$ into mutually orthogonal sectors $\mathcal{H}^\lambda$, so that

$$A \,|_{\mathcal{H}^\lambda} = (\dim \, D^\lambda) \; \Delta^\lambda, \tag{17}$$

$$B \,|_{\mathcal{H}^\lambda} = (\dim \, \Delta^\lambda) \; D^\lambda, \tag{18}$$

where dim denotes the dimension of an appropriate irrep. In other words, each sector $\mathcal{H}^\lambda$ is the carrier space of (dim D^λ) copies of irrep Δ^λ, and, at the same time, of (dim Δ^λ) copies of D^λ. More specificly, the dual sets $SYT(\lambda)$ and $WT(\lambda, \tilde{n})$ of standard Young and Weyl tableaux of the shape λ, respectively, serve as appropriate basis and/or repetition labels for the dual irreps.

We proceed to describe the set $RC(\lambda)$, $\lambda \vdash N$, $|N| \leq n$, of rigged string configurations, the building blok for a complete labeling of eigenstates of the Heisenberg Hamiltonian. It can be introduced for an arbitrary single-node spin s (and thus an arbitrary n), but we restrict ourselves in the sequel to the case $s = 1/2$, so that $n = 2$, and thus $\lambda = (\lambda_1, \lambda_2) \equiv (N - r, r)$, with $r \leq N/2$ having the interpretation of the number of spin deviations from the ferromagnetic saturation state $| + +...+ >$ (in all states of the highest weight, that is for the total spin $S = N/2 - r$ equal to its z-projection $M = \sum_{j \in \tilde{N}} f(j)$). One can also interpret the integer r labeling the partition λ as the number of Bethe pseudoparticles which are hard-core indistinguishable particles, allowed to move on the finite crystal $\tilde{N}$ [15]. A *string configuration* is introduced in this case as a partition $\nu \vdash r$ of the integer r. Each part of this partition, or, equivalently, each row of the Young diagram of ν, is referred to as a *string* of the length l equal to the number of boxes in this row. A string configuration $\nu \vdash r$ satisfies thus the sum rule

$$\sum_l l m_l = r, \tag{19}$$

with m_l denoting the number of strings of the length l in a given eigenstate.

A complete characterization of an eigenstate with the string configuration ν is given by *rigging* which is realized by the set

$$\mathcal{L} = \{L_\alpha^j | l = 1, 2, ..., \alpha = 1, 2, ..., m_l\}, \tag{20}$$

where L_α^j are nonnegative integers, one integer prescribed to each string (l, α) of ν, according to some quantization rules, which determine the range of rigging. These rules can be expressed in terms of *pyramids* and *holes* [5-8]. A string of the length l can be presented as a pyramid, that is isosceles triangle with the base 2l and the height l, with vertices on a square lattice with axes $(j, 2S)$. Such a pyramid can be looked at as the sequence of $2l$ consecutive nodes j of the crystal $\tilde{N}$, first l of them occupied by the state $i = 1$, or the spin "+", and the last l - by $i = 2$, or by Bethe pseudoparticles. Various pyramids can move with the discrete step 1 on the crystal $\tilde{N}$ and/or on the slopes of larger pyramids, with some constraints resulting from the hard-core and undistinguishability of Bethe pseudoparticles. Essentially, a smaller pyramid can move on each of the two slopes of a higher one until their heights coincide [7]. Simple combinatoric considerations yield that the range P^l for the strings of the length l in the string configuration ν is given by

$$P^l = N - 2Q^l, \tag{21}$$

where Q^l is the number of boxes in the first l *columns* of the Young diagram of the string configuration ν. Quantum numbers of riggings have the range determined by

$$0 \leq L_\alpha^l \leq P^l, \quad L_\alpha^l \leq L_{\alpha'}^l \text{ for } \alpha < \alpha', \tag{22}$$

and thus the total number of admissible riggings of the string configuration ν is

$$|z(\nu)| = \prod_l \binom{P^l + m_l}{m_l} \tag{23}$$

P^l is interpreted as the number of *holes* of the length l and the rigging L_α^β as the number of admissible moves of the string (l, α) to its leftmost position in the crystal $\tilde{N}$.

The third basis mentioned in the introduction is

$$b_{\text{eigen}} = \bigcup_{\lambda \vdash N |\lambda| \leq 2} WT(\lambda, \tilde{2}) \times RC(\lambda), \tag{24}$$

where

$$RC(\lambda) \equiv RC(N-r,r) = \bigcup_{\nu \vdash r} z(\nu), \tag{25}$$

and $z(\nu)$ is the set of all riggings of the string configuration ν.

3 Bijections of Robinson-Schensted and of Kerov-Kirillow-Reshetikhin.

We have presented three different bases in the space $\mathcal{H}$ of all quantum states of the Heisenberg magnet: the set $\tilde{n}^{\tilde{N}}$ of magnetic configurations f, the set b^W_{irr} of all pairs (t,y) of Weyl and Young tableaux of the duality of Weyl, and the set b_{eigen} of elements $(t, \nu\mathcal{L})$ involving rigged string configurations which classify exact solutions of Bethe. Elements of all these states are some combinatorial objects buildt from the two alphabets, the nodes $\tilde{N}$ and the spins $\tilde{n}$, according to certain combinatoric rules which reflect transformational properties of the space $\mathcal{H}$ under the actions of the symmetric group Σ_N and the unitary group U(n). Here we make some short remarks on two important bijections which arrange these sets into the chain

$$\tilde{n}^{\tilde{N}} \xrightarrow{RS} b^W_{irr} \xrightarrow{KKR} b_{eigen}, \tag{26}$$

or

$$f \longmapsto (t,y) \longmapsto (t, \nu\mathcal{L}) \tag{27}$$

(cf. Fig. 1 for an example).

$$f = \begin{array}{ccccccc} j= & 1 & 2 & 3 & 4 & 5 & 6 \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ i= & - & - & + & - & + & + \end{array} \xrightarrow{RS} \left(\begin{array}{c} P(f) = \begin{array}{|c|c|c|} \hline + & + & + \\ \hline - & - & - \\ \hline \end{array} = t \\ \\ Q(f) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} = y \end{array} \right) \xrightarrow{KKR} \left(\begin{array}{|c|c|} \hline 0 & \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{c} 0 \\ 2 \end{array} \right) = \nu\mathcal{L}$$

$a)$ $b)$ $c)$

Fig. 1. An Example of combinatorial objects and mappings for $N = 6$ nodes and $r = 3$ spin deviations: a) a magnetic configuration $f \in \tilde{2}^{\tilde{6}}$, b) the corresponding pair $(P(f), Q(f))$ of Weyl and Young tableaux under the RS bijection, c) the corresponding rigged string configuration $\nu\mathcal{L}$ under the KKR bijection; the latter consists of two strings, of the length 2 and 1, the range of rigging 0 and 2, and the actual rigging 0 and 1, respectively.

Here $RS : \tilde{n}^{\tilde{N}} \longrightarrow b^W_{irr}$ is the famous Robinson-Schensted algorithm [10-14], and $KKR : b^W_{irr} \longrightarrow b_{eigen}$ is a bijection introduced by Kerov, Kirillov and Reshetikhin [5-8]. We do not aim to describe them here (in fact, both bijections are usually presented in a larger and more general context of a monoid on the alphabet $\tilde{n}$, exceeding thus the framework of the Heisenberg magnet), but only point out for two features which might be important in magnetism and nanoscopic physics. Firstly, we stress that these bijections are purely combinatoric, which means that they do not involve any features outside the initial and target sets. In particular, both bijections explore the natural linear order of the alphabet $\tilde{n}$ of spins (the linear order of the alphabet $\tilde{N}$ of nodes is already explored in the construction of the set $\tilde{n}^{\tilde{N}}$). Thus tiny solutions of tremendously nonlinear Bethe equations are completely classified by mere combinatoric sorting procedures on finite sets. Secondly both bijections are complete on orbits

$$O_\mu = \{ f \circ \sigma^{-1} | \sigma \in \Sigma_N \} \tag{28}$$

of the symmetric group Σ_N on the set $\tilde{n}^{\tilde{N}}$, with the orbit label $\mu = (\mu_1, \mu_2, ..., \mu_n)$ being the weight of the configuration $f \in \tilde{n}^{\tilde{N}}$ (that is, μ_i is the number of nodes occupied by the spin $i \in \tilde{n}$ in configuration f), as well as on the sectors $\mathcal{H}^\lambda$ of the quantum space $\mathcal{H}$, In particular, we have the sum rule

$$\dim\Delta^\lambda = \sum_\nu |z(\nu)|. \tag{29}$$

which is a precise tool for predicting and checking the number of various string configurations, that is, various types of bound states of the exact BA solutions. For example, in the case of $N = 8$, $n = 2$, one readily checks that among $\binom{8}{4} = 70$ states with zero total magnetization there are $\dim\Delta^{\{4^2\}} = 14$ highest weight eigenstates, with possible string configurations $\nu = \{4\}, \{3\ 1\}, \{2^2\}, \{2\ 1^2\}$ and $\{1^4\}$, containing 1,5,1,6, and 1 rigged eigenstates, respectively. In other words, among various five configurations of strings of coupled magnons, three occur only once (a four-string, two two-strings, and four single magnons), the composite three-string and a single magnon - five times, and the composite two-string with two single magnons - six times. Such calculations are quite far from a combinatorial explosion for many nanoscopic sizes.

4 Conclusions

We have pointed out that exact solutions of BA for nanoscopic systems can be classified within a relatively simple combinatorial scheme which involves three sets: magnetic configurations, pairs of Weyl and Young tableaux, and rigged string configurations. Bijections of Robinson-Schensted and Kerov-Kirillov-Reshetikhin between these sets allow us to perform this classification just by combinatorial sorting procedures involving comparisons of letters of two alphabets, nodes and spins, which enter explicit solution of highly nonlinear Bethe equations for nanoscopic rings.

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