Magnetic interpretation of the Robinson - Schensted - Knuth algorithm

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Abstract

Combinatorial aspects of the Robinson-Schensted-Knuth (RSK) algorithm have been discused in the context of a Heisenberg magnetic ring with N nodes, each with the spin s. Each magnetic configuration acquires a natural interpretation as a word of the length N in the alphabet of spins, consisting of $n = 2s + 1$ letters. We demonstrate that the construction of n-tuple cover of the ring, with a separate copy for each letter of the alphabet of spins, allows for a transparent determination of maximal length of non-decreasing subwords. Moreover, it yields completeness of the RSK correspondence in classification of the irreducible basis of the Weyl duality between actions of unitary and symmetric groups in the space spanned on all magnetic configurations.

1 Introduction

The famous algorithm of Robinson [1], Schensted [2], and Knuth [3] associates, in a combinatorially unique way, each word f in an alphabet A with a pair $(P(f), Q(f))$ of standard Weyl and Young tableaux [4, 8, 9]. This algorithm, which is well adjusted to an abstract combinatorics on words in terms of non-decreasing sequences, plactic monoid, Schutzenberger involution etc ..., exhibits also an immediate relation to the description of the kinematics of the linear Heisenberg magnetic ring. Usefulness of the Robinson-Schensted-Knuth (RSK) algorithm in the theory of the Heisenberg magnet stems essentially from two facts. Firstly, a magnetic configuration of a linear ring with N nodes, each node with the spin s has an obvious interpretation as a word f of the length N in the alphabet A of single-node spin projections, such that $|A| = n = 2s + 1$. Secondly, the pair $(P(f), Q(f))$, prescribed to this configuration, labels an element of an irreducible basis of the space $\mathcal{H}$ of all quantum states of the magnet along the duality of Weyl [5, 6, 7, 10] between the actions of the symmetric group Σ_N and the unitary group $U(n)$ in $\mathcal{H}$.

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The quantum relationship between these two objects, a magnetic configuration f and its RSK image $(P(f), Q(f))$ is, hovever, not purely combinatorial, for reason of the linear structure of the space $\mathcal{H}$. Namely, an element $(P(f), Q(f))$ of the irreducible basis of the Weyl duality is *not* a single magnetic configuration but, rather a wave packet (a linear superposition) of several such configurations [11, 12, 13].

In the present paper we discuss combinatorial aspects of magnetic interpretation of the RSK algorithm, in order to make transparent both the mathematical status and the physical meaning of relevant objects, and to point out an interpretation of the RSK algorithm within the framework of the quantum-mechanical description of the Heisenberg model of magnetism. The linear aspects of this interpretation are presented in a accompaning paper [15].

2 Combinatorics of the Heisenberg magnet

The quantum description of the Heisenberg magnet starts with some elementary notions rooted in combinatorics. A one-dimensional magnetic Heisenberg ring consists of the set

$$\tilde{N} = \{j = 1, 2, ..., N\} \tag{1}$$

of N nodes of a crystal, and can be modelled as a regular orbit of the cyclic group C_N. It can be also looked at as *the alphabet of nodes*. Each node carries an n-dimensional unitary space h - a carrier space of the irreducible representation (irrep) D^s of the unitary unimodular group $SU(2)$, such that $2s + 1 = n$, and an orthonormal basis in h labelled by the set

$$\tilde{n} = \{i = 1, 2, ..., n\}, \tag{2}$$

so that $m = i - s - 1$ is a z-projection of the single-node spin s. We refer herefrom to the set $\tilde{n}$ as to *the alphabet of spins*. We refer to each mapping $f : \tilde{N} \to \tilde{n}$ as *a magnetic configuration*, and denote by

$$\tilde{n}^{\tilde{N}} = \{f : \tilde{N} \to \tilde{n}\} \tag{3}$$

the set of all magnetic confugurations on the ring $\tilde{N}$. A magnetic configuration $f \in \tilde{n}^{\tilde{N}}$ can be written in a form

$$f = f(1)f(2)\cdots f(N), \tag{4}$$

and treated therefore as a word of the length N in the alphabet of spins. In this way, the set $\tilde{n}^{\tilde{N}}$ constitues a sector in the free monoid $\tilde{n}^*$, consisting of all words in the alphabet of spins, with concatenation of words as monomial multiplication. We have

$$\tilde{n}^* = \bigcup_{N=0}^{\infty} \tilde{n}^{\tilde{N}}. \tag{5}$$

The Weyl duality [10] originates from two actions, $A : \Sigma_N \times \mathcal{H} \to \mathcal{H}$ and $B : U(n) \times \mathcal{H} \to \mathcal{H}$, of the symmetric group Σ_N and the unitary group $U(n)$ in the space

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}} \tag{6}$$

of all quantum states of the magnet. It is the linear closure of the set of all magnetic configurations with the inner product $(\ ,\ ) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$(f, f') = \delta_{ff'}, \quad f \in \tilde{n}^{\tilde{N}}. \tag{7}$$

A detail presentation of this duality is given in [15]. Here we recall that the space $\mathcal{H}$ of all quantum states of the magnet decomposes into sectors

$$\mathcal{H} = \sum_{\lambda \vdash N, |\lambda| \leq n} \oplus\ \mathcal{H}^\lambda, \tag{8}$$

which are labelled by partitions λ of N into not more than n nonzero parts, and each sector is factorised as

$$\mathcal{H}^\lambda = U^\lambda \otimes V^\lambda, \tag{9}$$

where U^λ and V^λ are carrier spaces, respectively, of an irreducible representation (irrep) D^λ of $U(n)$ and Δ^λ of Σ_N. In order to introduce combinatoric definitions of irreducible bases in spaces U^λ and V^λ, and thus in $\mathcal{H}^\lambda$ and $\mathcal{H}$, we need the notion of a standard tableau. A *Weyl tableau* t of the shape λ is any semistandard filling of boxes of the Young diagram corresponding to the partition λ by the alphabet $\tilde{n}$ of spins, i.e., the entries of boxes weakly increase along rows and strictly increasy along columns. A *Young tableau* of the shape λ is any standard filling of such a diagram by the alphabet $\tilde{N}$ of nodes, i.e., entries strictly increase in both rows and columns, so that the filling is bijective. We denote the set of all Weyl tableaux of the shape λ in the alphabet $\tilde{n}$ by $WT(\lambda, \tilde{n})$, and the set of all Young tableaux of the shape λ by $SYT(\lambda)$. In this way

$$\mathcal{H} = lc_{\mathbb{C}} WT(\lambda, \tilde{n}) \times SYT(\lambda), \tag{10}$$

so that pairs $(t, y) \in WT(\lambda, \tilde{n}) \times SYT(\lambda)$ of tableaux form *the irreducible basis of the Weyl duality* in the sector space $\mathcal{H}^\lambda$.

Each vector (λ, t, y) of the irreducible basis in $\mathcal{H}^\lambda$ ($\lambda = sh\, t = sh\, y$) is evidently a wave packet composed as a linear superposition of several magnetic configurations, as is explained in [15]. The RSK procedure associates each triad (λ, t, y) with a *single* magnetic configuration, on purely combinatoric grounds, presented in the following sections.

3 Application of the Robinson-Schensted-Knuth algorithm to magnetic configurations

Application of the RSK algorithm to a magnetic configuration $f : \tilde{N} \to \tilde{n}$, treated as a word of the length N in the alphabet $\tilde{n}$ of spins, consists essentially on a sorting of consecutive single-node states $f(j)$ on the chain $\tilde{N}$. The aim of the sorting procedure is to make transparent how to extract some non-decreasing subwords, arranged according to non-increasing lengths. The sequence of such subwords defines, in a combinatorially unique way, a Weyl tableau $P(f)$, with rows formed from these subwords. Correspondingly, the sequence of sorting the single-particle states, defines an associated Young tableau $Q(f)$ in the alphabet $\tilde{N}$ of nodes of the chain.

The mapping P of Schensted is conveniently defined in the context of the free monoid $\tilde{n}^*$. Such a description is well adopted to the structure of Young and Weyl tableau which reflect the branching rules under the chain $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_N \cdots$ of symmetric groups and combines with the branching rules for the chain $U(1) \subset U(2) \subset \cdots \subset U(n)$ of unitary groups. We proceed to present appropriate combinatorial notions.

One calls each pair $f(j)f(j+1)$ of consecutive letters in a word f a *descent* if $f(j+1) < f(j)$ (in the natural order of the set $\tilde{n}$). If (j_1, j_1+1) and (j_2, j_2+1) denote localisations of consecutive descents in the word f, then the subword

$$v = f(j_1+1)\, f(j_1+2)\, \cdots\, f(j_2) \tag{11}$$

is clearly non-decreasing; such a word is referred to as a *row*. In this way, each word $f \in \tilde{n}^*$ is uniquely decomposed into the product of rows. We write it in the form

$$f = v_l v_{l-1} \cdots v_2 v_1, \tag{12}$$

in order that the notation corresponds to conventions for Weyl tableaux. Let v and v' be two rows. By the definition, the row v' *dominates* the row v if the length of v' is not longer than that of v, that is

$$\begin{array}{lll} v = & a_1 a_2 \cdots a_r \cdots a_s & a_i \in \tilde{n} \\ v' = & b_1\, b_2 \cdots b_r, & b_i \in \tilde{n}, \end{array} \tag{13}$$

and, moreover,

$$b_i > a_i \;\text{ for }\; i = 1, 2, \ldots, r. \tag{14}$$

A word $f \in \tilde{n}^*$ is called a *tableau* if each of its row v in the decomposition (12) dominates the row v_{i-1}, $i = 2, 3, \ldots, l$. For a tableau f, the sequence of lengths

$$\lambda = (|v_1|, |v_2|, \ldots, |v_l|) \vdash N \tag{15}$$

constitutes a partition of the word f of the length N. When inserting each row v_i into the Young diagram of the shape λ, one gets a Weyl tableau from the set $WT(\lambda, \tilde{n})$. Clearly, the notion of dominance of rows serves as a rephrasing of well known conditions of semistandardness of a Weyl tableau in the context of the monoid $\tilde{n}^*$.

Evidently, only selected words in the monoid $\tilde{n}^*$ correspond to semistandard tableaux. Let $WT(\tilde{n}) \subset \tilde{n}^*$ be the set of all tableaux in the monoid $\tilde{n}^*$. The RSK algorithm defines the mapping $P : \tilde{n}^* \to WT(\tilde{n})$, which prescribes a tableau $P(f) \in WT(\tilde{n})$ to each word $f \in \tilde{n}^*$. This prescription is performed by an appropriate rearrangement of letters of f by means of a recursive procedure, in which each step consists of an *insertion* of a letter $x = f(j)$ (the consecutive j-th letter of the word f) into the tableau t^{j-1}, constructed already from the initial subword $f(1)f(2)\cdots f(j-1)$. Formally, we have

$$P\big(f(1)f(2)\cdots f(j)\big) = INS\Big(f(j), P\big(f(1)f(2)\cdots f(j-1)\big)\Big), \tag{16}$$

where $INS(x,t)$ denotes the insertion of a letter $x \in \tilde{n}$ into a tableau $t \in WT(\tilde{n})$. A transparent definition of $INS(x,t)$ can be given in two steps. Firstly, the insertion of a letter x into a row

$$v = a_1 a_2 \cdots a_l, \ \ a_i \leq a_{i+1} \ \text{ for } \ i = 1, 2, ..., l-1, \tag{17}$$

consists in putting x at the end of v when $x \geq a_l$, or in replacing the left most letter $a_r > x$ by x, and placing the bumped letter a_r in the second row, that is,

$$INS(x, v) = \left\{ \begin{array}{ll} \boxed{\ v \quad\ \ } \boxed{x} & \text{for } \ x \geq a_r \\ \\ \begin{array}{l} \boxed{\ v' \qquad\quad\ } \\ \boxed{a_r} \end{array} & \text{for } \ x < a_r \end{array} \right. , v \in Row(\tilde{n}), \ \ x \in \tilde{n}, \tag{18}$$

where $Row(\tilde{n}) \subset WT(\tilde{n})$ denotes the set of all rows in the monoid $\tilde{n}^*$, and

$$v' = a_1\, a_2 \cdots a_{r-l}\, x\, a_{r+1} \cdots a_l, \ a_{r-l} \leq x < a_{r+1} \tag{19}$$

is the row obtained from v by replacing the letter a_r by x. Secondly, for an arbitrary $t \in WT(\tilde{n})$, written as

$$t = v_l v_{l-1} \cdots v_2 v_1, \ v_i \in Row(\tilde{n}), \ i = 1, 2, ..., l, \tag{20}$$

$INS(x,t)$ is a new Weyl tableau, obtained as follows: x is put to the first row v_1 according to Eq. (18), either at its end (in which case the procedure terminates), or somewhere inside. In the latter case, the bumped letter a_r should be inserted by means of Eq. (18) in the second row v_2, etc. The procedure terminates after a finite number of steps. In general, we have

$$P(fx) = P\big(P(f)x\big) = INS\big(x, P(f)\big), \tag{21}$$

which completes the definition of the Schensted mapping P. This mapping defines the following equivalence relation on the monoid $\tilde{n}^*$

$$f \sim_P f' \Longleftrightarrow P(f) = P(f'), \ f, f' \in \tilde{n}^*. \tag{22}$$

The quotient of this relation,

$$\tilde{n}^*/P \cong WT(\tilde{n}) \tag{23}$$

is called the plactic monoid [8]. In this respect, each Weyl tableau $t \in WT(\tilde{n})$ is an element of the plactic monoid $\tilde{n}^*/P$, and consists of all words which yield the same Weyl tableau, i.e.

$$t \cong \{f \in \tilde{n}^* | P(f) = t\}. \tag{24}$$

One proves that each relation $f \sim f'$ can be generated by a sequence of local transpositions of adjacent letters, given by *Knuth relations*

$$\begin{array}{ll} a_1a_3a_2 \sim_P a_3a_1a_2 & \text{for } a_1 \le a_2 < a_3, \\ a_2a_1a_3 \sim_P a_2a_3a_1 & \text{for } a_1 < a_2 \le a_3, \end{array} \tag{25}$$

which result from an application of the definition of P to the case $N = 3$.

It is clear that the RSK procedure can be looked at as a recurrence construction of the Weyl tableau $t = P(f)$ by a consecutive insertion of a single letter of the word f. At the j-th step one arrives at the tableau

$$t^j = P\big(f(1)f(2)\cdots f(j)\big), \tag{26}$$

so that the whole procedure defines the chain of tableaux

$$t^{(1)} \subset t^{(2)} \subset \cdots \subset t^{(N)} = P(f). \tag{27}$$

Each difference

$$t^{(j)} \setminus t^{(j-1)} = t_{\alpha(j)\beta(j)} \tag{28}$$

defines a single box $t_{\alpha\beta}$ of the resulting tableau $t = P(f)$. By insertion of the letter $j \in \tilde{N}$ from the alphabet of nodes into the box $t_{\alpha(j)\beta(j)}$, one obtains the Young tableau

$$y = Q(f) \in SYT(\lambda) \tag{29}$$

of the same shape λ as $P(f)$, that is

$$shQ(f) = shP(f) = \lambda. \tag{30}$$

In this way, one defines the dual mapping $Q : \tilde{n}^* \longmapsto SYT_n$ of the monoid $\tilde{n}^*$ onto the set SYT_n of all standard Young tableaux of shapes given by partitions $\lambda \vdash N, N = 0, 1, 2, \ldots$, into not more than n parts. This mapping defines another equivalence relation on the monoid $\tilde{n}^*$, namely

$$f \sim_Q f' \Leftrightarrow Q(f) = Q(f'), \;\; f, f' \in \tilde{n}^*, \tag{31}$$

and the quotient of this relation is

$$\tilde{n}^*/Q \cong SYT_n. \tag{32}$$

Elements of this quotient set are standard Young tableaux. Thus each tableau y can be looked at as the set

$$y \cong \big\{ f \in \tilde{n}^* \,\big|\, Q(f) = y \big\}, \tag{33}$$

referred sometimes to as a coplactic class [8].

Alternatively, the RSK algorithm can be realised by a recursive construction of the Young standard tableau $Q(f)$, accompanied by a parallel building of the Weyl tableau $P(f)$. To this end, it is convenient to present each magnetic configuration f in a form of a rectangular $n \times N$ table $m(f)$, with elements defined by

$$m_{ij}(f) = \delta_{i,f(j)}, \;\; i \in \tilde{n}, \, j \in \tilde{N}. \tag{34}$$

Thus, the matrix $m(f)$ has entries from the set $\{0, 1\}$, with clear meaning of its rows $m_{i\bullet}(f)$ and columns $m_{j\bullet}(f)$. The row $m_{i\bullet}(f)$ consist of N elements, with μ_i "ones" pointing out of those nodes which are ocuppied by the spin $i \in \tilde{n}$. The column $m_{\bullet j}(f)$ has a single "one" which shows the spin $f(j) = i \in \tilde{n}$ by which the node j is occupied. Clearly, the mapping $f \mapsto m(f)$ establishes a bijection between each orbit $\mathcal{O}_\mu$ of the symmetric group Σ_N on $\tilde{n}^{\tilde{N}}$ (cf. Eq. (34)) and the set $M_{n\times N}(\mu, 1^N)$ of all $\{0, 1\}$ - matrices of the shape $n \times N$ with the row sums

$$\sum_{j \in \tilde{N}} m_{ij} = \mu_i, \;\; i \in \tilde{n}, \tag{35}$$

and columns sums

$$\sum_{i \in \tilde{n}} m_{ij} = 1, \;\; j \in \tilde{N}, \tag{36}$$

forming the composition μ and partition $\{1^N\}$ of N, respectively.

Let

$$z(f) = m_{1\bullet} m_{2\bullet} \cdots m_{n\bullet} \;\; (m_{1\bullet} \ldots) \tag{37}$$

be the sequence of consecutive rows of the matrix $m(f)$ with the last sequence $m_{n\bullet}$ joined, moreover, to the first one m_1. The sequence $z(f)$ can be interpreted as an n - tuple *cover* of the magnetic ring $\tilde{N}$, with each "winding" $m_{i\bullet}$ being a faithful copy of the ring $\tilde{N}$, with "ones" at nodes occupied by the spin i and "zeros" elsewhere.

In other words, the cover $(\tilde{n}, \tilde{N}, f)$ of a given magnetic configuration consists of n copies $m_{i\bullet}$ of the ring $\tilde{N}$. Each copy $m_{i\bullet}$ has distinguished all those nodes of the ring $\tilde{N}$ which are occupied by the spin $i \in \tilde{n}$. Evidently, such a cover provides a transparent tool for sorting procedures, in particular for extracting non-decreasing and/or strictly increasing subwords of the word f, which correspond respectively to the rows and/or columns of $\lambda = sh\, P(f) = sh\, Q(f)$, in accordance with the combinatorial theorem of Greene [14]. Such a construction defines a natural cyclic order modulo Nn. A consecutive counting of "ones" in the cover $z(f)$ defines a reordering of nodes of the ring N, given by a permutation

$$\sigma(f) = \begin{pmatrix} j \\ \sigma(f, j) \end{pmatrix}, \;\; j \in \tilde{N}, \tag{38}$$

where $\sigma(f, j)$ is the "old" label of the node which acquires the label $j \in \tilde{N}$ in the ordering of the cover $z(f)$. In other words, if

$$j = \sum_{i'=1}^{i} \mu_{i'} + l, \;\; l \leq \mu_{i+1}, \tag{39}$$

then $\sigma(f,j)$ is the localisation of the l - th node with the spin $i+1$ in the magnetic configuration $f \in \mathcal{O}_\mu$. By putting

$$word(\sigma) = \sigma(1)\sigma(2)\cdots\sigma(N), \;\; \sigma \in \Sigma_N, \tag{40}$$

one associates a word in the alphabet $\tilde{N}$ of nodes, of the length N and weight $\{1^N\}$, with each permutation $\sigma \in \Sigma_N$. The $word(\sigma(f))$ serves as the starting point of an equivalent iterative RSK procedure such that at the j - th step one gets the Young tableau

$$y^{(j)} = P\big(\sigma(f,1)\ldots\sigma(f,j)\big), \tag{41}$$

with the Schensted mapping P applied here to nodes rather than spins. The corresponding Weyl tableau $t = P(f)$ is build correspondingly, by adding at the j -th step the box

$$t_{\alpha(j)\beta(j)} = f\big(\sigma(i,j)\big) \tag{42}$$

in the place determined by the corresponding box

$$y^{(j)} \setminus y^{(j-1)} = y_{\alpha(j)\beta(j)}, \;\; j \in \tilde{N}, \tag{43}$$

in the chain of Young tableaux. The word

$$std(f) = word(\big(\sigma(f)^{-1}\big)) \tag{44}$$

is known as the *standardization* of the word f [8].

4 Final remarks and conclusions

We have demonstrated here the meaning of the RSK algorithm in the context of the Heisenberg model of magnetism. Based on purely combinatorial considerations, such as the sorting of spins at consecutive nodes of the magnetic ring N, one obtains a unique bijection

$$RSK : \tilde{n}^{\tilde{N}} \longrightarrow \bigcup_{\lambda \vdash N, \;\; |\lambda| \leq n} WT(\lambda, \tilde{n}) \times SYT(\lambda) \tag{45}$$

between each magnetic configuration f and the triad $(\lambda, P(f), Q(f))$, with

$$\lambda = sh\, P(f) = sh\, Q(f), \;\; p(f) \in WT(\lambda, \tilde{n}), \;\; Q(f) \in SYT(\lambda). \tag{46}$$

Equation (45) implies the sum rule

$$n^N = \sum_{\lambda \vdash N, \;\; |\lambda| \leq n} dim\, D^\lambda \, dim\, \Delta^\lambda, \tag{47}$$

in which each summand is $dim\, \mathcal{H}^\lambda$. Thus the RSK correspondence allows the use of the set $\tilde{n}^{\tilde{N}}$ of magnetic configuration as a complete set of labels for the irreducible basis of the Weyl duality.

The structure of the n-tuple cover $(\tilde{n}, \tilde{N}, f)$ for each magnetic configuration allows us also to establish a bijection which associates each orbit $\mathcal{O}_\mu$ of the action of the symmetric group Σ_N on the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations with the set $M_{n\times N}(\mu, 1^N)$ of all $n \times N$ $0-1$ matrices with the row sums μ and column sums 1^N. This bijection provides an important link between purely combinatorial considerations on lengths of nondecreasing subwords and representation theory. This link is expressed in the Kostka decompositon

$$R^\mu = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} \Delta^\lambda \tag{48}$$

of the transitive representation $R^\mu = A\big|_{\mathcal{O}_\mu}$ into irreps Δ^λ, with multiplicities being Kostka numbers $K_{\lambda\mu}$. The corresponding sum rule reads

$$\frac{N!}{\prod_{i\in\tilde{n}} \mu_i!} = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} dim\, \Delta^\lambda = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} K_{\lambda\{1^N\}}. \tag{49}$$

This decomposition implies that the image $RSK(\mathcal{O}_\mu)$ of any orbit $\mathcal{O}_\mu$ of Σ_N spans *complete* Σ_N-irreducible subspaces in the space $\mathcal{H}$. The second paper [15] reports an explicit construction of such subspaces, together with appropriate orthogonality and completness relations.

It is worth observing here that the mathematical setting of the whole infinite monoid $\widetilde{n}^*$; instead of a finite set $\tilde{n}^{\tilde{N}}$, constituting its N-th sector, proves to be very convenient due to the fact that it is compatible with the infinite chain $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_j \subset \cdots$ of symmetric groups. This chain explains also the meaning of structure and growth of standard Young and Weyl tableaux, which are, in fact, immediately related to the RSK correspondence. Just a mere definition of the Schensted function $P : \widetilde{n}^* \to WT(\tilde{n})$ involves consecutive insertions of letters into tableaux. Thus the RSK procedure proves to be natural not only in such purely combinatorial problems as selection of longest non-decreasing subwords, but also in classification of irreducible bases of the Weyl duality and related quantum-mechanical considerations.

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