Kinematics of the Heisenberg chain and irreducible bases of the Weyl duality

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Abstract

Quantum kinematics of the linear Heisenberg ring, consisting of N crystal nodes, each with spin s, is presented in terms of the Weyl duality between actions of the symmetric and unitary groups in the space of quantum states of the magnet. This space is spanned on magnetic configurations which gives rise to an application of the combinatorial Robinson - Schensted - Knuth algorithm for a unique classification of irreducible basis of the duality of Weyl in terms of a pair of tableaux: a standard Young tableau in the alphabet of nodes, accompanied by a semistandard Weyl tableau in the alphabet of spins. Similarities and distinctions between various group-theoretic and combinatorial objects are discussed within the context of the magnetic interpretation. In particular, the role of the spectrum of Jucys-Murphy operators in the clasification and construction of magnetic eigenstates corresponding to Young tableaux is illustrated.

1 Introduction

The scheme of Weyl duality [1] between actions of the symmetric group Σ_N and the unitary group $U(n)$ in the N-th tensor power space $h^{\otimes N}$ of a single-particle n-dimensional space h plays an essential role in a quantum-mechanical description of the kinematics of multiparticle states in atoms [2, 3, 6, 7], nuclei [4, 9, 10, 11], elementary particles [12, 13], quantum chemistry [14, 15] and solid state [5, 8]. Here, we aim to point out its role in the kinematics of the Heisenberg model of magnetism. We confine our attention to a finite magnetic chain of N nodes because of its importance with respect to the Bethe Ansatz solution [16] and associated topics related to exactly solvable models [17, 18], although most of our considerations are applicable to a magnetic cluster with an arbitrary geometric shape.

The Heisenberg model of magnetism is especially well adapted to the Weyl duality scheme in its full generality since the arena for kinematics of the model spans exactly the *whole* tensor power space $h^{\otimes N}$, without limitations imposed

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by, say, statistics, to some specific subspaces (bosons, fermions, paraparticles, two-row or two-column Young diagrams for spin-orbit spaces in quantum chemistry, four-row or four-column diagrams for spin-isospin in nuclear shells, etc.). Within the Heisenberg model, there are no such "superselection rules", and *all* states within $h^{\otimes N}$ are physically meaningful. Here we aim to interpret this phenomenon in terms of positions. To this purpose, we associate with each magnetic configuration a "position" of the system. Clearly, the set of all magnetic configurations forms an orthonormal basis in $h^{\otimes N}$, and thus all linear superpositions, or "wave packets", are admissible states of the system. We point out that the irreducible basis of the duality of Weyl can be naturally interpreted in terms of "axial positions", with the corresponding quantum numbers determined as eigenvalues of a complete set of operators, referred to as Jucys-Murphy operators [20, 21, 22]. The corresponding eigenstates span some Σ_N-irreducible subspaces, which, in a particular case of the single-node spin $s = 1/2$, correspond to a definite value S of the total spin of the magnet.

2 Kinematics of the Heisenberg ring and the duality of Weyl

We consider a linear magnetic Heisenberg ring, consisting of N nodes, each with the spin s. Let

$$\tilde{N} = \{j = 1, 2, ..., N\} \tag{1}$$

and

$$\tilde{n} = \{i = 1, 2, ..., n\}, \;\; n = 2s + 1, \tag{2}$$

be, respectively, the set of nodes of the magnetic ring, and the set of all single-node spin projections, the set of spins in short. The set $\tilde{N}$, referred also to as *the alphabet of nodes*, constitutes a regular orbit of the cyclic group C_N, and serves as the defining set for the symmetric group Σ_N, such that $C_N \subset \Sigma_N$. The set $\tilde{n}$, *the alphabet of spins*, provides an orthonormal basis for the carrier space h of a single-node spin s, such that

$$h = lc_{\mathbb{C}}\tilde{n}, \;\; dim\, h = n = 2s + 1, \tag{3}$$

that is, h is the linear closure of the set $\tilde{n}$ over the field $\mathbb{C}$ of complex numbers. Each mapping $f : \tilde{N} \longmapsto \tilde{n}$, written in a form

$$|f\rangle = |i_1, i_2, ..., i_N >, \;\; i_j \in \tilde{n}, \;\; j \in \tilde{N}, \tag{4}$$

defines *a magnetic configuration* on the ring $\tilde{N}$, and constitutes a *word* of length N in the alphabet of spins.

The set

$$\tilde{n}^{\tilde{N}} = \{f : \tilde{N} \longrightarrow \tilde{n}\} \tag{5}$$

of all such magnetic configurations provides an orthonormal basis of the space $\mathcal{H}$ of all quantum states of the magnet, so that

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}} \cong \prod_{j \in \tilde{N}} \otimes\, h_j, \tag{6}$$

where h_j is a faithful copy of the linear unitary space h, attributed to the node $j \in \tilde{N}$.

We recall that, according to the scheme of Weyl duality [1], the space $\mathcal{H}$ is a scene of two dual actions, $A : \Sigma_N \times \mathcal{H} \to \mathcal{H}$ and $B : U(n) \times \mathcal{H} \to \mathcal{H}$, determined on the basis vectors $\tilde{n}^{\tilde{N}}$ by the formulas

$$A(\sigma) = \begin{pmatrix} f \\ f \circ \sigma^{-1} \end{pmatrix},\ \ f \in \tilde{n}^{\tilde{N}},\ \ \sigma \in \Sigma_N, \tag{7}$$

$$B(u) = \begin{pmatrix} f \\ uf \end{pmatrix},\ \ f \in \tilde{n}^{\tilde{N}},\ \ u \in U(n), \tag{8}$$

where $f \circ \sigma^{-1}$ is the composition of mappings $f : \tilde{N} \longmapsto \tilde{n}$ and $\sigma^{-1} : \tilde{N} \longrightarrow \tilde{N}$, so that

$$|f\rangle = |i_1, ..., i_N\rangle \overset{\sigma^{-1}}{\longmapsto} |i_{\sigma^{-1}(1)}, ..., i_{\sigma^{-1}(N)}\rangle = |f \circ \sigma^{-1}\rangle \tag{9}$$

for

$$\sigma^{-1} = \begin{pmatrix} 1 & \ldots & N \\ \sigma^{-1}(1) & \ldots & \sigma^{-1}(N) \end{pmatrix} \in \Sigma_N, \tag{10}$$

the linear action uf reads in more detail as

$$u\, f \equiv B(u)\, |i_1 \ldots i_N\rangle = \sum_{i'_1, ..., i'_N \in \tilde{n}} u_{i'_1 i_1} \ldots u_{i'_N i_N}\, |i'_1, ..., i'_N\rangle \tag{11}$$

for

$$u = \begin{pmatrix} u_{11} & \ldots & u_{1n} \\ \vdots & \ldots & \vdots \\ u_{n1} & \ldots & u_{nn} \end{pmatrix} \in U(n), \tag{12}$$

and these formulas are extended from the basis $\tilde{n}^{\tilde{N}}$ to the whole space $\mathcal{H}$ by linearity. The most important quantum -mechanical observation is that the two actions mutually commute, that is,

$$[A(\sigma), B(u)] = 0,\ \ \sigma \in \Sigma_N,\ \ u \in U(n), \tag{13}$$

despite the fact that both dual groups are, for $N > 2, n > 1$, highly noncommutative. It implies compatibility of appropriate quantities related to both groups in the spirit of the Heisenberg uncertainty principle: these quantities "can be measured simultanously". A maximal system of such compatible (commuting) observables is realised in an irreducible basis in the space $\mathcal{H}$, adapted to the symmetry of both dual groups. We proceed to describe this scheme in more detail.

Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \ \ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0, \ \sum_{i \in \tilde{n}} \lambda_i = N, \tag{14}$$

be a partition of the integer N into not more than n parts. This partition serves as the label of the irreducible representation (irrep) Δ^λ of the symmetric group Σ_N and, at the same time, as the label of the irrep D^λ of the unitary group $U(n)$. The corresponding decompositions of both dual actions into irreps read

$$A = \sum_{\lambda \vdash N, |\lambda| \leq n} m(A, \lambda) \ \Delta^\lambda \tag{15}$$

and

$$B = \sum_{\lambda \vdash N, |\lambda| \leq n} m(B, \lambda) \ D^\lambda \tag{16}$$

where appropriate multiplicities satisfy duality relations

$$m(A, \lambda) = dim\, D^\lambda, \tag{17}$$

$$m(B, \lambda) = dim\, \Delta^\lambda, \tag{18}$$

with dimensions expressible in terms of famous hooklength formulas [23]. The symbol $\lambda \vdash N$ in Eqs. (15)-(16) denotes a partition of N, and $|\lambda|$ is the numbers of non-zero parts ($\lambda_i > 0$) of this partition. Correspondingly, the space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \sum_{\lambda \vdash N, |\lambda| \leq n} \oplus \ \mathcal{H}^\lambda \tag{19}$$

into sectors $\mathcal{H}^\lambda$. Each sector carries $m(A, \lambda)$ copies of Δ^λ - irreducible subspaces of the symmetric group Σ_N, which can be labelled conveniently by an irreducible basis of the irrep D^λ of the unitary group $U(n)$, since according to Eq. (17), the multiplicity of Δ^λ is just equal to dimension of D^λ. Dually, the same sector carries $m(B, \lambda)$ copies of D^λ - irreducible subspaces of the unitary group $U(n)$, labelled by an irreducible basis of Δ^λ of Σ_N. This is an effect of commutativity (13), or of the fact that the actions A and B in $\mathcal{H}$ mutually centralize. As a result, one introduces for each sector $\mathcal{H}^\lambda$ an irreducible standard basis $|\lambda\, t\, y\,\rangle$ with the properties

$$A(\sigma) |\lambda\, t\, y\,\rangle = \sum_{y' \in \tilde{\Delta}^\lambda} \Delta^\lambda_{y' y}(\sigma)\, |\lambda\, t\, y'\,\rangle, \ \ \sigma \in \Sigma_N, \tag{20}$$

and

$$B(u) |\lambda\, t\, y\,\rangle = \sum_{t' \in \tilde{D}^\lambda} D^\lambda_{t' t}(u)\, |\lambda\, t'\, y\,\rangle, \ \ u \in U(n), \tag{21}$$

where $\Delta^\lambda_{y' y}(\sigma)$ and $D^\lambda_{t' t}(u)$ are standard matrices for the irrep specified by the partition λ for the group Σ_N and $U(n)$, respectively, with $\tilde{\Delta}^\lambda$ and $\tilde{D}^\lambda$ being the corresponding sets of irreducible bases. In other words, each sector $\mathcal{H}^\lambda$ is factorised as

$$\mathcal{H}^\lambda = U^\lambda \otimes V^\lambda, \tag{22}$$

with

$$U^\lambda = lc_{\mathbb{C}} \tilde{D}^\lambda, \;\; V^\lambda = lc_{\mathbb{C}} \tilde{\Delta}^\lambda \tag{23}$$

being standard $U(n)$ and Σ_N - irreducible modules, respectively. In this way, the irreducible basis label for each action ($t \in \tilde{D}^\lambda$ for B and $y \in \tilde{\Delta}^\lambda$ for A) serves also as the repetition label for the dual action, in accordance with the general spirit of the duality of Weyl. At this point, the bases $\widetilde{D}^\lambda$ and $\widetilde{\Delta}^\lambda$ can be chosen arbitrarily, as any (orthonormal and complete) sets of vectors in the space U^λ and V^λ, respectively. In the next section we specify these irreducible bases to be consistent with the Young orthogonal form of irreps Δ^λ [21, 22, 24].

In fact, the Robinson-Schensted-Knuth (RSK) [25, 26, 27, 28] algorithm provides a way of labeling the irreducible scheme of the duality of Weyl by magnetic configurations $f \in \tilde{n}^{\tilde{N}}$ in a combinatorially unique way. It is now obvious, however, that each irreducible state $|\lambda\, t\, y\,\rangle$ is a definite linear superposition (a wave packet) of a number of magnetic configurations and should not be confused with a *single* magnetic configuration, even if the RSK procedure yields a combinatorially unique bijective correspondence $RSK : f \mapsto |\lambda\, t\, y\rangle$. We demonstrate the construction of such a wave packet in the following sections, in terms of famous Kostka matrices [29] at the level of bases.

3 Kostka matrices at the level of irreducible bases

Let us first consider the action A of the symmetric group Σ_N as a purely permutational representation $A : \Sigma_N \times \tilde{n}^{\tilde{N}} \to \tilde{n}^{\tilde{N}}$. This action decomposes the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations of the ring into orbits

$$\mathcal{O}_\mu = \{ f \circ \sigma^{-1} |\, \sigma \in \Sigma_N \} \tag{24}$$

labelled by *weights* (or compositions)

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n), \;\; \sum_{i \in \tilde{n}} \mu_i = N, \tag{25}$$

so that

$$\mu_i = |\{ i_j = i \,|\, j \in \tilde{N} \}|, \;\; i \in \tilde{n} \tag{26}$$

is the occupation number for the single-node state $i \in \tilde{n}$ for any $f \in \mathcal{O}_\mu$. Such an orbit carries the transitive representation $R^{\Sigma_N : \Sigma^\mu}$, with the stabiliser

$$\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \ldots \times \Sigma_{\mu_n} \tag{27}$$

being the Young subgroup for an appropriate $f \in \mathcal{O}_\mu$. A *stratum* of the action A, i.e. the set of all orbits with the same stabiliser class, is defined by the sequence

$$\nu = (\nu_0, \nu_1, \nu_2, \dots, \nu_N), \;\; \sum_{l=0}^{N} l\nu_l = N, \tag{28}$$

determined by lengths l present in the weight μ, that is,

$$\nu_l = |\{\mu_i = l \,|\, i \in \tilde{n}\}|, \;\; l \in \tilde{N}, \tag{29}$$

denotes the number of parts of the length l in the weight μ, and

$$\nu_0 = n - \sum_{l \in \tilde{n}} \nu_l \tag{30}$$

is the number of those single-node states $i \in \tilde{n}$, which do not occupy any node in the magnetic configurations entering the orbit $\mathcal{O}_\mu$. The stratification of the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations under the action A of the symmetric group reads therefore

$$\tilde{n}^{\tilde{N}}/A = \bigcup_{\nu} S(\nu), \tag{31}$$

where the stratum $S(\nu)$ consists of

$$|S(\nu)| = \frac{n!}{\nu_0! \prod_{l \in \tilde{N}} \nu_l!} \tag{32}$$

permutationally equivalent orbits $\mathcal{O}_\mu$, each with

$$|\mathcal{O}_\mu| = \frac{N!}{\prod_{i \in \tilde{n}} \mu_i!} \tag{33}$$

magnetic configurations. The total number of orbits is

$$|\tilde{n}^{\tilde{N}}/A| = \sum_{\nu} |S(\nu)| = \binom{N+n-1}{N}, \tag{34}$$

and the total number of magnetic configurations satisfies the sum rule

$$n^N = \sum_{\nu} |S(\nu)| \circ |\mathcal{O}_\mu|, \tag{35}$$

where $\mathcal{O}_\mu$ is an orbit entering the stratum $S(\nu)$.

Next, we take into account the linear structure of the action A of the symmetric group Σ_N in the space $\mathcal{H}$. In terms of transitive representations, this action decomposes as

$$A \cong \sum_{\nu} \oplus \, |S(\nu)| \, R^{\Sigma_N : \Sigma^\nu}, \tag{36}$$

and each transitive constituent is subject to the Kostka decomposition into irreps

$$R^{\Sigma_N : \Sigma^\nu} \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu} \; \Delta^\lambda, \tag{37}$$

where multiplicities $K_{\lambda\,\mu}$ constitute the Kostka matrix for the group Σ_N, and $\lambda \trianglerighteq \mu$ denotes the dominance order in the partially ordered set of partitions (see, e.g., [29]) (in fact, one has to choose that orbit $\mathcal{O}_\mu \in S_\nu$ for which the weight μ is a partition, that is, satisfies $\mu_1 \geqslant \mu_1 \geqslant \ldots \geqslant \mu_n$; such a weight exists in each stratum $S(\nu)$).

Now, we are ready to introduce a Σ_N-irreducible basis in $\mathcal{H}$ exhibiting the decomposition (36)-(37). Obviously such a basis has the form

$$|\mu\,\lambda\,t\,y\rangle = \sum_{f\in\mathcal{O}_\mu} \left[\begin{array}{ccc} \mu & \lambda & t \\ f & y & \end{array}\right] |f\rangle, \tag{38}$$

that is, a wave packet, composed as a linear superposition of several magnetic configurations f from the orbit $\mathcal{O}_\mu$. We refer herefrom to coefficients of this wave packet,

$$K^{t\,y,\,f}_{\lambda\,\mu} = \left[\begin{array}{ccc} \mu & \lambda & t \\ f & y & \end{array}\right] \tag{39}$$

as elements of the Kostka matrix at the level of bases, which is justified by a comparison with Eq. (37) (the level of irreps). The whole matrix is labelled by the partition μ, has the size $|\mathcal{O}_\mu| \times |\mathcal{O}_\mu|$ (cf. Eq. (33)), its rows are labelled by $f \in \mathcal{O}_\mu$, and its columns-by triads $(\lambda\,t\,y)$. In the following sections we identify these triads with the outcome of the RSK algorithm.

The standard basis $\tilde{\Delta}^\lambda$ for the irrep Δ^λ of the symmetric group Σ_N consist of the set $\mathrm{SYT}(\lambda)$ of all standard Young tableaux of the shape λ. By the combinatorial definition, a standard Young tableau $y \in \mathrm{SYT}(\lambda)$ is the Young diagram corresponding to the partition λ, filled in bijectively by the alphabet $\tilde{N}$ of nodes, such that each row from left to right and each column from top to bottom constitutes a strictly increasing sequence. Thus, in particular, the upper leftmost box has to be filled in by the node $j = 1$. In other words, we assume hereafter that

$$\widetilde{\Delta}^\lambda = SYT(\lambda). \tag{40}$$

We feel obliged to stress at this point that the combinatorial definition of the set $SYT(\lambda)$ of standard Young tableaux given above does *not* yet specify a standard basis for the irreducible space V^λ, or, equivalently, matrix elements $\Delta^\lambda_{y'y}(\sigma)$, $\sigma \in \Sigma_N$, of Eq.(20). A tableau $y \in SYT(\lambda)$ is in a common use in the most of literature to *label* a basis vector $v_y \in V^\lambda$ but this vector remains undetermined until appropriate requirements concerning its transformational properties under the action of the group Σ_N are formulated [29]. In particular, one has to distinguish between the basis of polytabloids [24] and the Young orthogonal basis [21, 22, 24], even if both are conventionally labelled by the same set $SYT(\lambda)$. The former basis is not orthogonal (in particular, $\Delta^\lambda_{y'y}(\sigma) \in \mathbb{Z}$ are integers in this presentation), whereas the latter can be made consistent with the unitary structure of the space $\mathcal{H}$ of quantum states of the magnet, fixed by physical requirements. We demonstrate this in the next section, by use of Jucys-Murphy operators.

4 Jucys-Murphy operators

A complete set of basis states in the Σ_N-irreducible space V^λ is provided by simultaneous eigenvectors of the set of commuting Hermitan operators, referred hereafter as Jucys-Murphy operators [19, 20] (cf. also [16, 21, 22, 30, 31]), along the general quantum-mechanical recipe of Dirac [32]. They are defined by the formula

$$\hat{M}_j = \sum_{j'=1}^{j-1} (j', j), \quad j = 2, 3, \ldots, N, \tag{41}$$

where $(j', j) \in \mathbb{C}(\Sigma_N)$ denotes the transposition of nodes j and j' in $\tilde{N}$. Thus the operator $\hat{M}_j$ is the sum of all transpositions of the node $j \in \tilde{N}$ with preceding nodes $j' < j$. Jucys-Murphy operators are Hermitian in the standard inner product of the group algebra $\mathbb{C}(\Sigma_N)$, mutually commute, i.e.

$$[\hat{M}_j, \hat{M}_{j'}] = 0, \tag{42}$$

and span a maximal Abelian subalgebra in $\mathbb{C}(\Sigma_N)$. Their common eigenvalues within the space V^λ are labelled by Young tableaux $y \in \mathrm{SYT}(\lambda)$. The eigenproblem for Jucys-Murphy operators reads

$$\hat{M}_j \,|\, y\rangle = m_j(y) \,|\, y\rangle, \quad y \in SYT(\lambda), \tag{43}$$

with

$$m_j(y) = c_j(y) - r_j(y), \tag{44}$$

where $c_j(y)$ and $r_j(y)$ denotes the column and row, respectively, of the location of the node $j \in \tilde{N} \setminus \{1\}$ in the Young tableau y.

Thus each eigenvalue $m_j(y)$ of the Jucys-Murphy operator $\hat{M}_j, \ j \in \tilde{N} \setminus \{1\}$, has the meaning of *axial distance* of location of the corresponding node j in the tableau y from the main diagonal of the Young diagram $\lambda = \mathrm{sh}\, y$, the *shape* of y. The eigenvector $|y\,\rangle \in V^\lambda$ is thus specified up to a normalising and phase factor by the sequence (vector) of quantum numbers - eigenvalues of $\hat{M}_j$'s

$$\vec{m} = (m_1(y), m_2(y), \ldots m_N(y)), \quad y \in SYT(\lambda) \tag{45}$$

with $m_1(y) = 0$, which are axial locations of all nodes $j \in \tilde{N}$ of the ring in the Young tableau y. Jucys-Murphy operators $\hat{M}_j$ thus constitute a set of quantum-mechanical observables which are mutually compatible and form a complete set for classification of basis eigenstates in the space V^λ.

Jucys-Murphy operators determine in this way standard basis vectors $|\, y\rangle$ for the Σ_N - irreducible space V^λ with the accuracy to a phase. The latter is fixed by the requirement that all non-vanishing off-diagonal matrix elements $\langle y\,|(j, j+1)|\, y'\rangle, j = 1, 2, \ldots N-1$ of elementary transpositions $(j, j+1)$ should be positive. It yields the well known orthogonal Young form of the irreps Δ^λ of Σ_N, which can be expressed in terms of Jucys-Murphy operators as follows. Let $y \in SYT(\lambda)$ be a Young tableau, $\Sigma_2^{(j,j+1)} \subset \Sigma_N$ be the subgroup of the symmetric group Σ_N, generated by the single elementary transposition $(j, j+1), j = 1, 2, \ldots N-1$, so that

$$V^\lambda_{(j,j+1)} = \mathbb{C}(\Sigma_2^{(j,j+1)})\,|\,y\rangle \subset V^\lambda \tag{46}$$

is a subspace of the space V^λ. Let, moreover

$$\beta = \frac{1}{m_{j+1}(y) - m_j(y)} \tag{47}$$

be the inverse of the axial distance between boxes occupied by nodes j and $j+1$ in the Young tableau y. Then one has three cases:
(i) $\beta = 1$, $dim V^\lambda_{(j,j+1)} = 1$, and

$$(j, j+1)\,|\,y\rangle = |\,y\rangle, \tag{48}$$

(ii) $\beta = -1$, $dim V^\lambda_{(j,j+1)} = 1$, and

$$(j, j+1)\,|\,y\rangle = -|\,y\rangle, \tag{49}$$

(iii) $-1 < \beta < 1$, $dim V^\lambda_{(j,j+1)} = 2$, and the space $V^\lambda_{(j,j+1)}$ is spanned in this case by two vectors, $|\,y\rangle$ and $|\,y'\rangle$, where y' is obtained from y by the interchange of entries j and $j+1$ (with all other entries unchanged; one proves that y' is standard). One obtains

$$\Delta^\lambda\Big((j, j+1)\Big)\Big|_{V^\lambda_{(j,j+1)}} = \begin{array}{c|cc} & y & y' \\ \hline y & \beta & \sqrt{1-\beta^2} \\ y' & \sqrt{1-\beta^2} & -\beta \end{array}, \tag{50}$$

which defines the Young orthogonal form of Δ^λ completely (in virtue of the fact that elementary transpositions generate the whole symmetric group Σ_N).

The sum of all Jucys-Murphy operators

$$\hat{C}^{(2)} = \sum_{j \in \tilde{N}} \hat{M}_j = \sum_{1 \le j' < j \le N} (j', j) \tag{51}$$

constitutes the sum over all transpositions in Σ_N, i.e. the operator of the class of transpositions, which belongs to the center of the group algebra $\mathbb{C}(\Sigma_N)$. Thus its eigenvalue is the same within each space V^λ, and is given by

$$C^{(2)}_\lambda = \frac{1}{2} \sum_{i \in \tilde{n}} \lambda_i(\lambda_i - 2i + 1), \tag{52}$$

which is just the sum of axial distances of all boxes of the Young diagram λ. It is worth mentioning that for $n = 2$ the famous Dirac identity, which in our notation reads

$$A(j, j+1) = \frac{1}{2}(I + \vec{\sigma}_j \otimes \vec{\sigma}_{j+1}), \tag{53}$$

where I is the identity operator in $\mathcal{H}$ and σ_j is the Pauli vector matrix for the node $j \in \tilde{N}$ in $\mathcal{H}$, associates this eigenvalue with the total spin S of the magnet by the formula

$$A(\hat{C}^2) = \frac{N(N-4)}{4} I + \hat{\vec{S}}^2. \tag{54}$$

.

Therefore, in the case $s = 1/2$, or $n = 2$, each Σ_N-irreducible subspace V^λ, $\lambda = \{N - r', r'\}$, of the linear span of any orbit $\mathcal{O}_\mu$, $\mu = \{N - r, r\}$, $r' \leq r \leq N/2$, carries a definite total spin $S = N/2 - r'$, determined by the shape of λ, and its z - projection $M = N/2 - r$, defined by the unique Weyl tableau

$$t = \begin{array}{l} \overbrace{\boxed{+}\boxed{\quad\cdots\quad}\boxed{+}}^{N-r}\overbrace{\boxed{-}\boxed{\cdots}\boxed{-}}^{r-r'} \\ \underbrace{\boxed{-}\boxed{\cdots}\boxed{-}}_{r'} \end{array} \tag{55}$$

Various states $|\lambda\, t\, y\rangle \in lc_{\mathbb{C}}\mathcal{O}_\mu$ differ mutually by the Young tableau y, that is, by various distributions of axial distance over the nodes of the chain $\tilde{N}$, which is encoded in appropriate eigenvalues of Jucys-Murphy operators. The total sum of axial distances, given by the eigenvalue of the operator $A(\hat{C}^2)$, is constant within the manifold defined by λ and t, and varies for different manifolds accordingly to the shape of λ.

5 Properties of Kostka matrices at the level of bases

The definition (39) of Kostka matrices at the level of bases implies a ladder construction following the combinatorial growth of a Weyl tableau t by attachement of consecutive nodes $j = 1, 2, \ldots N$ to already constructed state of $j - 1$ nodes. Thus we have

$$\begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix} =$$

$$\begin{bmatrix} \{1\} & \{1\} & \lambda_{12} \\ f(1) & f(2) & t_{12} \end{bmatrix} \begin{bmatrix} \lambda_{12} & \{1\} & \lambda_{123} \\ t_{12} & f(3) & t_{123} \end{bmatrix} \cdots \begin{bmatrix} \lambda_{1\ldots N-1} & \{1\} & \lambda \\ t_{1\ldots N-1} & f(N) & t \end{bmatrix}. \tag{56}$$

Here the symbol

$$\begin{bmatrix} \lambda_{1\ldots j-1} & \{1\} & \lambda_{1\ldots j} \\ t_{1\ldots j-1} & f(j) & t_{1\ldots j} \end{bmatrix} \tag{57}$$

denotes the coupling coefficient at the j-th stage of growth, corresponding to the Littlewood-Richardson decomposition

$$D^{\lambda_{1\ldots j-1}} \otimes D^{\{1\}} = \sum_{\lambda_{1\ldots j}} \oplus\ D^{\lambda_{1\ldots j}} \tag{58}$$

for the unitary group $U(n)$. This decomposition is multiplicity-free (one attaches a *single* box $\{1\}$ to the Young diagram $\lambda_{1\ldots j-1}$ in all places admissible by the requirements of standardness), and thus the coupling coefficients in the right-hand side of (56) do not involve any repetition label for arguments in each of their upper row.

The bottom row of the coupling coefficient (57) exhibits the standard bases corresponding to irreps presented in the first row. Bases for N constituent single-node irreps $\{1\}$ are understood as Weyl tableaux with a single box, filled in by $f(j) \in \tilde{n}$ at the j-th stage of growth. In this way, the magnetic configuration f which enters the Kostka matrix element at the level of bases in the left hand-side of (56) defines irreducible bases for constituent irreps in the right hand-side.

Intermediate irreps $\lambda_{1\ldots j}$ and the corresponding bases $t_{1\ldots j},\ \ j \in \tilde{N}\ \ (\lambda_{1\ldots N} = \lambda,\ \ t_{1\ldots N} = t)$ are defined in terms of quantum numbers $\lambda\, t\, y$ by means of the Robinson-Schensted-Knuth (RSK) correspondence [25, 26, 27, 28, 29] which associates with each $f' \in \tilde{n}^{\tilde{N}}$ a pair

$$(t, y) = (P(f'), Q(f')),\ \ f' \in \tilde{n}^{\tilde{N}}, \tag{59}$$

such that

$$sh\, t = sh\, y = \lambda. \tag{60}$$

We intend to give elsewhere [39] a thorough discussion of interpretation of RSK correspondence within the model of the Heisenberg magnet. Here we only mention that RSK provides an algorithm which codes uniquely the growth process presented in Eq. (56), so that the arguments of the intermediate and resultant irreps are determined by the triad

$$RSK(f') = (\lambda, t, y) \tag{61}$$

as follows. The intermediate irrep $\lambda_{1\ldots j}$ is given by

$$\lambda_{1\ldots j} = sh\, y^{(j)}, \tag{62}$$

where $y^{(j)}$ is obtained from the Young tableau y by deleting all letters $j' > j$ of the alphabet of nodes. The corresponding Weyl tableau $t_{1\ldots N}$ is obtained from t by consecutive deletion of letters $f'(N),\ \ f'(N-1),\ \ \ldots,\ \ f'(j+1)$ of the word $f' \in \tilde{n}^{\tilde{N}}$ given by Eq.(61). Deletion of the last letter, $f'(N)$, of the word f' from the Weyl tableau t consists in the following steps. (i) Select the letter $t_{\alpha,\beta} \in \tilde{n}$, corresponding to the letter $N = y_{\alpha,\beta}$ in the Young tableau t, and remove the box (α, β) from t. (ii) Insert the selected letter $t_{\alpha,\beta}$ in the row $\alpha - 1$ of the tableau t in the place of the leftmost letter larger than $t_{\alpha,\beta}$, that is

$$t_{\alpha-1,\beta'} > t_{\alpha,\beta},\ \ t_{\alpha-1,\beta'-1} \leq t_{\alpha,\beta}. \tag{63}$$

(iii) Continue the procedure with $t_{\alpha-1,\beta'}$, etc ..., until removing a definite letter from the first row of t; the latter letter is just $f'(N)$. Next, one has to remove the letter $f'(N-1)$ from the Weyl tableau $t_{1\ldots N-1}$ according to $y^{(N-1)}$, then

$f'(N-2)$ etc., until reading any $t_{1\ldots j}$. In this way, all symbols entering the formula (56) for elements of Kostka matrix at the level of bases are defined in terms of two magnetic configurations, f and f'. The former is related to the initial basis of the space $\mathcal{H}$ of quantum states of the magnet, and the latter - to the irreducible basis of the duality of Weyl. Explicit relation of the latter magnetic configuration, f', to the Weyl triad (λ, t, y) is provided by the RSK algorithm.

Equation (56) allows us to evaluate the Kostka matrices at the level of bases, once the coupling coefficients (57) are known. Properties of these coefficients have been described extensively by Louck [29-31] in terms of Maclaurin polynomials and labelled rooted trees within combinatorial aspects of representation theory of unitary groups. An extensive calculus allows us to evaluate each matrix recursively in terms of $SU(2)$ Clebsch-Gordan coefficients [32-33]. Here we do not consider this subject, and only give in Tables 1, 2 and 3 simple illustrative examples.

By the definition (56), the linear span $lc_{\mathbb{C}}\mathcal{O}_\mu$ of any orbit $\mathcal{O}_\mu$ of the symmetric group Σ_N on the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations carries a Σ_N-complete irreducible basis of the duality of Weyl. Kostka matrices at the level of bases satisfy therefore the following orthogonality and completeness relations

$$\begin{array}{c} \sum_{f\in\mathcal{O}_\mu} \left[\begin{array}{ccc} \mu & \lambda & t \\ f & y & \end{array}\right]^* \left[\begin{array}{ccc} \mu & \lambda' & t' \\ f & y' & \end{array}\right] = \delta_{\lambda\lambda'}\delta_{tt'}\delta_{yy'}, \\ \sum_{\lambda\trianglerighteq\mu}\sum_{t\in WT(\lambda,\tilde{n}}\sum_{y\in SYT(\lambda)} \left[\begin{array}{ccc} \mu & \lambda & t \\ f & y & \end{array}\right]^* \left[\begin{array}{ccc} \mu & \lambda & t \\ f' & y & \end{array}\right] = \delta_{f,f'}. \end{array} \tag{64}$$

The RSK algorithm provides a combinatorial ground for these relations: each orbit $\mathcal{O}_\mu$ of the action A of the symmetric group is "combinatorially complete" with the meaning that it yields a complete labelling for the irreducible basis of the Weyl duality.

6 Final remarks and conclusions

We have presented here the kinematics of the Heisenberg model of a magnet. Quantum kinematics, i.e., the classification of states of a magnet, can be transparently described in terms of Weyl duality between the single constituent - the $n = 2s+1$-dimensional spin space h, and the collection of N nodes, arranged geometrically into a ring $\tilde{N}$.The total space $\mathcal{H}$ of quantum states of the magnet decomposes into sectors $\mathcal{H}^\lambda$, classified by partitions λ of N into not more than n parts. A complete classification of irreducible bases in each sector $\mathcal{H}^\lambda$ is given by the set $SYT(\lambda)\times WT(\lambda,\tilde{n})$, consisting of pairs (t,y) of standard Weyl and Young tableaux, in the alphabet $\tilde{n}$ of spins and $\tilde{N}$ of nodes, respectively, both of the shape given by the partition λ. The Weyl tableau t yields the information of composition and statistical properties of composite spins, whereas the Young tableau y encodes spatial distribution of spins over the ring N.

The initial basis of $\mathcal{H}$, i.e. the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations can be interpreted, from (6), as a "classical configuration space", or the set of all

	λ	$\{5\}$	$\{41\}$			
	t	1 2 3 4 5	1 3 4 5 / 2	1 2 4 5 / 3	1 2 3 5 / 4	1 2 3 4 / 5
f						
$+++--$		$\frac{1}{\sqrt{10}}$	0	0	$\frac{-1}{2}$	$-\frac{\sqrt(15)}{10}$
$-+++-$		$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{6}$	$-\frac{\sqrt(15)}{10}$
$+-++-$		$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{6}$	$-\frac{\sqrt(15)}{10}$
$++-+-$		$\frac{1}{\sqrt{10}}$	0	$\frac{-\sqrt(2)}{3}$	$\frac{1}{6}$	$-\frac{\sqrt(15)}{10}$
$++--+$		$\frac{1}{\sqrt{10}}$	0	$\frac{-\sqrt(2)}{3}$	$\frac{-1}{3}$	$\frac{1}{\sqrt{15}}$
$-+-++$		$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{6}}$	$\frac{-1}{3\sqrt{2}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{15}}$
$-++-+$		$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$	$\frac{-1}{3}$	$\frac{1}{\sqrt{15}}$
$--+++$		$\frac{1}{\sqrt{10}}$	0	$\frac{\sqrt(2)}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{15}}$
$+-+-+$		$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$	$\frac{-1}{3}$	$\frac{1}{\sqrt{15}}$
$+--++$		$\frac{1}{\sqrt{10}}$	$\frac{-1}{\sqrt{6}}$	$\frac{-1}{3\sqrt{2}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{15}}$

	λ	$\{32\}$				
	t	1 3 5 / 2 4	1 3 4 / 2 5	1 2 5 / 3 4	1 2 4 / 3 5	1 2 3 / 4 5
f						
$+++--$		0	0	0	0	$\frac{\sqrt(2)}{2}$
$-+++-$		0	$\frac{-1}{\sqrt{3}}$	0	$\frac{-1}{3}$	$\frac{-1}{3\sqrt{2}}$
$+-++-$		0	$\frac{1}{\sqrt{3}}$	0	$\frac{-1}{3}$	$\frac{-1}{3\sqrt{2}}$
$++-+-$		0	0	0	$\frac{2}{3}$	$\frac{-1}{3\sqrt{2}}$
$++--+$		0	0	$\frac{1}{\sqrt{3}}$	$\frac{-1}{3}$	$\frac{-1}{3\sqrt{2}}$
$-+-++$		$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{6}$	$\frac{1}{3\sqrt{2}}$
$-++-+$		$\frac{-1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{-1}{3\sqrt{2}}$
$--+++$		0	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{3}$	$\frac{1}{3\sqrt{2}}$
$+-+-+$		$\frac{1}{2}$	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{-1}{3\sqrt{2}}$
$+--++$		$\frac{-1}{2}$	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{6}$	$\frac{1}{3\sqrt{2}}$

Table 1: Kostka matrix at the level of bases for $N = 5,\ \ n = 2,\ \ \mu = \{32\}$
.

	λ	$\{4\}$	$\{31\}$					
	t	$\begin{smallmatrix} a & a & b & c \end{smallmatrix}$	$\begin{smallmatrix} a & a & b \\ c & & \end{smallmatrix}$			$\begin{smallmatrix} a & a & c \\ b & & \end{smallmatrix}$		
	y	$\begin{smallmatrix} 1 & 2 & 3 & 4 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 3 & 4 \\ 2 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 & 4 \\ 3 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 & 3 \\ 4 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 3 & 4 \\ 2 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 & 4 \\ 3 & & \end{smallmatrix}$	$\begin{smallmatrix} 1 & 2 & 3 \\ 4 & & \end{smallmatrix}$
f								
$aabc$		$\frac{1}{2\sqrt(3)}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{-1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{-1}{2\sqrt(10)}$	$\frac{-1}{12}$	$\frac{\sqrt(2)}{3}$
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{-1}{\sqrt(10)}$	$\frac{-1}{3}$	$\frac{-\sqrt(2)}{6}$
$aabc$		$\frac{1}{2\sqrt(3)}$	0	0	$\frac{1}{2}$	$\frac{3}{2\sqrt(10)}$	$\frac{-1}{4}$	0
$aabc$		$\frac{1}{2\sqrt(3)}$	0	$\frac{\sqrt(2)}{3}$	$\frac{-1}{6}$	0	$\frac{1}{6}$	$\frac{\sqrt(2)}{3}$
$aabc$		$\frac{1}{2\sqrt(3)}$	0	0	$\frac{1}{2}$	$\frac{-3}{2\sqrt(10)}$	$\frac{-1}{4}$	0
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{-1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{1}{\sqrt(10)}$	$\frac{-1}{3}$	$\frac{-\sqrt(2)}{6}$
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{1}{2\sqrt(10)}$	$\frac{5}{12}$	$\frac{-\sqrt(2)}{6}$
$aabc$		$\frac{1}{2\sqrt(3)}$	0	$\frac{\sqrt(2)}{3}$	$\frac{-1}{6}$	$\frac{3}{2\sqrt(10)}$	$\frac{-1}{12}$	$\frac{-\sqrt(2)}{6}$
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{-1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{-1}{2\sqrt(10)}$	$\frac{5}{12}$	$\frac{-\sqrt(2)}{6}$
$aabc$		$\frac{1}{2\sqrt(3)}$	$\frac{1}{\sqrt(6)}$	$\frac{-\sqrt(2)}{6}$	$\frac{-1}{6}$	$\frac{1}{2\sqrt(10)}$	$\frac{-1}{12}$	$\frac{\sqrt(2)}{3}$
$aabc$		$\frac{1}{2\sqrt(3)}$	0	$\frac{\sqrt(2)}{3}$	$\frac{-1}{6}$	$\frac{-1}{2\sqrt(10)}$	$\frac{-1}{12}$	$\frac{-\sqrt(2)}{6}$

Table 2: Kostka matrix at a level of bases for $N = 4$, $n = 3$, $\mu = \{21\}$

.

λ	$\{2^2\}$		$\{21^2\}$		
t	$\begin{array}{\|c\|c\|}\hline a & a \\ \hline b & c \\ \hline\end{array}$		$\begin{array}{\|c\|c\|}\hline a & a \\ \hline b \\ \cline{1-1} c \\ \cline{1-1}\end{array}$		
y	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$
f					
$aabc$	$\frac{1}{\sqrt{(6)}}$	0	$\frac{1}{2}$	0	0
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{-1}{2\sqrt{(2)}}$	$\frac{1}{4}$	$\frac{\sqrt{(3)}}{4}$	0
$aabc$	$\frac{1}{\sqrt{(6)}}$	0	0	$\frac{\sqrt{(3)}}{6}$	$\frac{1}{\sqrt{(6)}}$
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{1}{2\sqrt{(2)}}$	$\frac{-1}{4}$	$\frac{\sqrt{(3)}}{4}$	0
$aabc$	$\frac{1}{\sqrt{(6)}}$	0	$\frac{-1}{2}$	0	0
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{-1}{2\sqrt{(2)}}$	$\frac{-1}{4}$	$\frac{-\sqrt{(3)}}{4}$	0
$aabc$	$\frac{1}{\sqrt{(6)}}$	0	0	$\frac{-\sqrt{(3)}}{6}$	$\frac{-1}{\sqrt{(6)}}$
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{-1}{2\sqrt{(2)}}$	$\frac{-1}{4}$	$\frac{1}{4\sqrt{(3)}}$	$\frac{-1}{\sqrt{(6)}}$
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{-1}{2\sqrt{(2)}}$	$\frac{1}{4}$	$\frac{-1}{4\sqrt{(3)}}$	$\frac{1}{\sqrt{(6)}}$
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{1}{2\sqrt{(2)}}$	$\frac{-1}{4}$	$\frac{-1}{4\sqrt{(3)}}$	$\frac{1}{\sqrt{(6)}}$
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{1}{2\sqrt{(2)}}$	$\frac{1}{4}$	$\frac{-\sqrt{(3)}}{4}$	0
$aabc$	$\frac{-1}{2\sqrt{(6)}}$	$\frac{1}{2\sqrt{(2)}}$	$\frac{1}{4}$	$\frac{1}{4\sqrt{(3)}}$	$\frac{-1}{\sqrt{(6)}}$

Table 3: Kostka matrix at a level of bases for $N = 4,\ n = 3,\ \mu = \{21\}$

.

"positions" of the "classical magnet", within the Schrodinger picture of quantum mechanics. Clearely, each magnetic configuration $f : \tilde{N} \to \tilde{n}$ is physically admissible, so that there are no "superselection rules" in $\mathcal{H}$.

We have pointed out here the role of Jucys-Murphy operators. They form a complete set of Hermitian commuting operators within the group algebra $\mathbb{C}(\Sigma_N)$ of the symmetric group and thus yield a complete quantum-mechanical classification of states in any Σ-irreducible subspace in $\mathcal{H}$. The corresponding eigenvalues measure "axial distances" of nodes of the magnet in a particular Σ_N-irreducible eigenstate, which express a maximum available information on statistical properties of quantum states under the action A of the symmetric group Σ_N. In this way, transformation to the irreducible basis of the duality of Weyl can be interpreted as a change of purely positional representation $\tilde{n}^{\tilde{N}}$ to "axial positions" $SYT(\lambda)$ in each sector $\mathcal{H}^{\lambda}$. Each such sector carries all states with the same total axial position, with individual states differing mutually by various distributions of axial distances over the nodes of the magnet. In particular, for $s = 1/2$, the partition λ defines (and is equivalent to) the total spin S of the magnet. Let us mention here that a "momentum representation" has been proposed [40] by decomposition of each orbit $\mathcal{O}_\mu$ of the symmetric group Σ_N into orbits of the cyclic group $C_N \subset \Sigma_N$, and Fourier transform on each C_N - orbit.

We have exhibited here the clear distinction between magnetic configuration which "clearly are " words of the length N in the alphabet $\tilde{n}$ of spins, and semistandard Weyl tableaux, which are sometimes also identified in mathematical literature with words (under appropriate rules for reading). Magnetic configurations form the initial basis for the space $\mathcal{H}$, whereas pairs (t,y) of tableaux provide the irreducible basis of the duality of Weyl. The distinction is expressed in terms of Kostka matrices at the level of bases, which are just transformations matrices between representations of "ordinary" and axial positions. In particular, this distinction is made clear in the context of RSK algorithm - a beautiful result which shows that each orbit $\mathcal{O}_\mu$ of the symmetric group on the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations is a unique and combinatorially complete source of quantum numbers for the Weyl duality scheme. It is worthwhile to point out that - due to RSK algorithm - there is *no summation* in the formula (56) for elements of Kostka matrices at the level of bases. Instead, we have there only a single term (which might be sometimes zero), the product of appropriate coupling coefficients, with initial quantum numbers f and resultant $f' \overset{RSK}{\longleftrightarrow} (\lambda, t, y)$ labelled by magnetic configurations. But evidently only f has an intrinsicly physical meaning of "position of a magnet", whereas f' denotes essentially an irreducible basis state of the Weyl duality, which is a quantum superposition of a number of magnetic configurations.

We deal here with the kinematics of a magnet, and thus leave aside dynamical problems like diagonalization of the Heisenberg Hamiltonian. Nevertheless, we'd like to point out that the commonly used exchange Hamiltonian, which is bilinear in spin operators and isotropic, is *not* invariant under the unitary group $U(n)$ for $n > 2$. Thus the Weyl duality is most suitable for the case $s = 2$, whereas dynamics for higher single-node spins s takes place also in the space $\mathcal{H}$ considered here, but one has to refine appropriately quantum calculations for exchange Hamiltonian, or to generalize the model to $U(n)$-invariant operators.

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